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A rigorous first order shear deformation theory (FSDT) is employed along with modified ABD parameters to analyze static and free vibration behavior of generally laminated beams and shafts. Different approaches for calculating composite beam stiffness parameters have been considered and the most accurate one that accounts for material couplings have been used to analyze static and free vibration behaviors of straight beams with different laminates and boundary conditions. In order to analyze curved beams, the term (1+z/R) is exactly integrated into ABD parameters formulation and an equivalent modulus of elasticity is used instead of traditional stiffness terms to account for both the deepness and material coupling of the beam structures. The model has been solved analytically for simply supported boundary conditions and the general differential quadrature (GDQ) technique has been used for other boundary conditions.

The results for deflection, moment resultants, and natural frequencies of straight and curved beams with different deepness ratio (often called depth ratio), slenderness ratio, lamination, and boundary conditions are compared with those obtained from accurate three dimensional finite element simulations using ANSYS. The results were in close proximity to three dimensional finite element results. The model is then applied to transverse vibration analysis of multi-span generally laminated composite shafts with a lumped mass using GDQ. The results for natural frequencies are compared to experimental and other analytical models as well as finite element simulation. The results in the present analyses were found accurate. Conclusively, it has been shown that when considering more accurate stiffness parameters, a First Order Shear Deformation Theory can accurately predict static and free vibration behaviors of composite beams and multi-span shafts of any deepness, lamination and boundary conditions.

Key words: Beam, Composite, Shaft, Vibration, Static, Deep

DEDICATION

I want to dedicate this dissertation to my dear wife, Maryam for her love and great help to me all over our life.

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CHAPTER I

INTRODUCTION

A structural element having one dimension many times greater than its other dimensions can be a rod, a bar, a column, or a beam. The definition actually depends on the loading conditions. A beam is a member mainly subjected to bending. The terms rod (or bar) and column are for those members that are mainly subjected to axial tension and compression, respectively. Beams are one of the fundamental structural or machine components and can be made of any material. Metallic beams as well as composite beams are used in different industries. Buildings, steel framed structures and bridges are examples of beam applications in civil engineering. In these applications, beams exist as structural elements or components supporting the whole structure. In addition, the whole structure can be modeled at a preliminary level as a beam. For example, a high rise building can be modeled as a cantilever beam, or a bridge modeled as a simply supported beam. In mechanical engineering, rotating shafts carrying pulleys and gears are examples of beams. In addition, frames in machines (e.g. a truck) are beams. Robotic arms in manufacturing are modeled as beams as well. In aerospace engineering, beams (curved and straight) are found in many areas of the plane or space vehicle. In addition, the whole wing of a plane is often modeled as a beam for some preliminary analysis. Innumerable other examples in these and other industries of beams exist.

Presently, the use of laminated composite beams, plates and shells in many engineering applications is rapidly expanding. Their higher strength and stiffness to weight ratios, the ability to tailor the design for specific purposes, and advances in the manufacturing methods, give them a competitive edge when compared with other engineering materials and lead to their extensive use.

Composite beams are lightweight structures that can be found in many diverse applications including aerospace, marine, medical equipment, automotive, construction as well as others industries. Their weight savings has been of great interest because of its general positive impact on durability, light weight, corrosion resistance as well as other attributes including fuel economy, noise, vibration and harshness (NVH).

On the other hand, composites have their own challenges in predicting failure, fatigue, or dynamic behaviors. The designers face much more complicated problems in designing such structures and they should broaden their knowledge to efficiently employ these benefits while considering all phenomena related to composites. Hence, the designers need comprehensive and easy to use formulas in any design problem.

This thesis is concerned with the development of the fundamental equations for the mechanics of laminated composite beams with different configurations (straight and curved), Laminates (cross-ply, angle-ply, …), cross sections (rectangular, tubular), and boundary conditions (clamped, simply supported, free, …) which can be useful for design engineers.

It should be mentioned here that the treatment presented in this thesis considers beams vibrating (or deforming) in their plane of curvature. Different formulas for the static and vibration analysis of laminated beams would be reviewed and compared with the three dimensional finite element method (3D FEM) analyses. 3D FEM analyses are based on the 3D elasticity theory. Thus, they are the most accurate analytical procedure to analyze the structure. However, 3D FEM is a very expensive procedure demanding both expensive machines and longer computer times if used for large scale structures. In addition, these structures have two dimensions smaller than the third. For these structures, beam models are very efficient provided they are built on accurate models and are verified against 3D analyses as is done in this thesis.

Two kinds of analyses will be performed in this thesis. First considered here is a static analysis where deflection and stress analyses for composite beams are performed. Second, dynamic analysis, where their natural frequencies and mode shapes are assessed. In many applications, deflection of the beam plays a key role in the design of the structure. For example, if an aircraft wing tip deflection becomes high, in addition to potential structural failure, it may deteriorate the wing aerodynamic performance. In this and other applications, beams can be subjected to dynamic loads. Imbalance in driveline shafts, combustion in crank shaft applications, wind on a bridge or a structure, earthquake loading on a bridge or a structure, impact load when a vehicle goes over a pump are all examples of possible dynamic loadings that beam structures can be exposed to. All of these loads and others can excite the vibration of the beam structure. This can cause durability concerns or discomfort because of the resulting noise and vibration.

Two classes of theories are developed for laminated beams. In the first class of theories, thin beams are studied where effects of shear deformation and rotary inertia are neglected. This class of theories will be referred to as thin beam theories or classical beam theories (CBT). This is typically accurate for thin beams and calculations of the most fundamental frequencies and is less accurate for thicker beams and higher frequencies. In the second class of theories, shear deformation and rotary inertia effects are considered. This class of theories will be referred to as thick beam theory or shear deformation beam theory (SDBT).

The following topics will be considered in this thesis. In chapter 2 a comprehensive literature review has been done on the analysis of laminated beams between 1989 and 2010. Chapter 3 deals with static and vibration analysis of generally laminated straight thick beams with different boundary conditions. In chapter 4 the analyses in chapter 3 is done for curved beams with different deepness ratios, slenderness ratios, laminates, and boundary conditions. Chapter 5 considers generally laminated thick multi-segment shafts with lumped mass that are solved by general differential quadrature (GDQ) method. Chapter 6 summarizes and concludes the research done in this thesis. The chapters are written in a way that any chapter can be read independently and has its own introduction, theory, results and discussion and conclusion.

The major contributions of this research are:

- Develop a consistent set of equations that take into consideration deepness, shear deformation (and rotary inertia), and general lamination of laminated composite curved beams

- Test the accuracy of the set of equations using exact solutions for simply supported boundaries and comparing results with 3D analyses for composite curved beams under static and dynamic loads
- Test the accuracy of the proposed set of equations for general boundary conditions using numerical methods
- Apply the proposed set of equations to solve practical problems. A multi-segmented composite shaft problem is solved for its vibrations.

CHAPTER II

LITERATURE REVIEW

Introduction

Laminated composite beams, plates and shells have been used in extensive applications in many engineering fields in recent decades. Structures composed of composite materials offer lower weight and higher strength and stiffness than those composed of most metallic materials. These advantages coupled with ability to tailor designs for specific reasons, give them a competitive edge when compared with normal engineering materials and has led to their extensive use. Composite beams, plates and/or shell components now constitute a large percentage of recent aerospace and submarine structures. They have found increasing use in areas such as automotive engineering and other applications. Composite beams act as lightweight load carrying structures in diverse applications from aerospace and automotive to construction industries.

Literature on composite beam research can be found in many conferences and journals. Kapania and Raciti [1] made a review on advances in analysis of laminated beams and plates vibration and wave propagation in 1989. Rosen [2] reviewed the research on static, dynamic, and stability analysis of pretwisted rods and beams in 1991. Chidamparam and Leissa [3] reviewed the published literature on the vibrations of curved bars, beams, rings and arches of arbitrary shape which lie in a plane in 1993. Also some books [4-7] discussed the analysis of composite beams, plates and shells.

This section focuses on the last two decades of research (1989 through 2010) done on the dynamic analysis of composite beams. The literature is reviewed while focusing on various aspects of research. We will first review the various beam theories that are being used in research in recent years. These include thin (or classical), thick (or shear deformation), and layerwise beam theories. Then different methods for solving equations of motion such as transfer matrix method, finite element method and others will be reviewed. Another aspect of research will be the use of smart materials, which include piezoelectric, shape memory materials. Complicating effects will be the final category that will be addressed. This will include viscoelastic effects, added mass, rotating beam, beams with imperfections and so on.

Beam Theories

Beams are generally three dimensional (3D) bodies bounded by four, relatively close surfaces. The 3D equations of elasticity are generally unnecessarily complicated when written for a beam. Researchers simplify such equations by making certain assumptions for particular applications. Almost all beam theories reduce the 3D elasticity problem into a one dimensional (1D) problem. There are two issues typically treated for 1D analysis of beams. The first problem is the issue of coupling and how to include the various couplings (stretching bending, bending twisting and others) that are ignored when reducing 3D equations to 1D. Since beams are 1D structural components, only the parameters along x axis are considered and from 18 parameters on 6*6 ABD stiffness matrix, just three terms (A_{11}, B_{11}, D_{11}) are used.

$$A_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k} \left[\left(h_k - h_{k-1} \right) \right] \tag{1}$$

$$B_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k} \frac{\left(h_k^2 - h_{k-1}^2 \right)}{2} \tag{2}$$

$$D_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k} \frac{\left(h_k^3 - h_{k-1}^3 \right)}{3} \tag{3}$$

Note that the beam width in the above terms is included in the definitions of these terms, while it is customary to leave this term out in general laminate analysis. To overcome coupling problem, instead of normal definition of A_{11}, B_{11}, and D_{11}, one may use equivalent stiffness parameters to find the equivalent ABD parameters.

The second problem is the inclusion of shear deformation and rotary inertia. In classical theories, shear deformation and rotary inertia terms are ignored and is generally accurate for thin beams; while in shear deformation theories, some of those terms are included and accuracy for thick beams increases. Shear deformation theories with higher order than one are treated in one section in this chapter titled higher order shear deformation theories.

Classical Beam Theory

If the beam thickness is less than 1/20 of the wavelength of the deformation mode, a classical beam theory (CBT) or Euler Bernoulli (EB) beam theory where shear deformation and rotary inertia are negligible, is generally acceptable for lower frequency determination. Qatu [8] and Qatu and Elsharkawi [9] used CBT to study vibration of straight and curved laminated beams. They used Ritz method for solving the equations of motion. Qatu and Iqbal [10] used CBT to solve the vibration of a cross-ply laminated composite driveshaft with an intermediate joint.

Mei [11] studied the effect of coupling between bending and torsional deformations on vibrations of composite EB beams from a wave vibration standpoint. The torsional mode is found unaffected by the material coupling only at low frequencies. The flexural modes were found to be affected by material coupling over the entire frequency band. Mei [12] also studied the local wave transmission and reflection characteristics at various discontinuities on composite beams. Gunda et al. [13] investigated large amplitude vibration of laminated composite beams with axially immovable ends with symmetric and asymmetric layup orientations. They used the CBT and solved the equations by Rayleigh–Ritz (R–R) method. Geometric nonlinearity of von-Karman type was considered which accounts for the membrane stretching action of the beam. Ecsedi and Dluhi [14] proposed a 1D mechanical model to analyze the static and dynamic feature of nonhomogeneous curved beams and closed rings. They expressed the equations of motion and the boundary conditions in terms of two kinematical variables. The first one was the radial displacement of cross-sections and the second one was the rotation of the cross-sections.

Shear Deformation Theories

The inclusion of shear deformation in the analysis of beams was first made by Timoshenko [15]. So, theories considering shear deformation are called as Timoshenko beam theories. The inclusion of shear deformation in the analysis happens in developing equations for displacement components. One approach for classification of beam theories is based on order of polynomial for approximation of displacements through the thickness. Suppose that the displacement u can be expressed as

$$u = u_0 + z\left[c_0 \frac{\partial w}{\partial x} + c_1 \phi(x,t)\right] + c_2 z^2 \psi\left(x,t\right) + c_3 \left(\frac{z}{h}\right)^3 \left(\frac{\partial w}{\partial x} + \phi(x,t)\right) \tag{4}$$

and ψ, ϕ are the rotation of a line element perpendicular to the original direction in the x and y direction, respectively. For this equation, special cases are defined as [6]

The Classical Beam Theory (CBT): c_0=-1, c_1=0, c_2=0, c_3=0

The First Order Shear Deformation Theory (FSDT): c_0=0, c_1=1, c_2=0, c_3=0

The Second Order Shear Deformation Theory (SSDT): c_0=0, c_1=1, c_2=1, c_3=0

The Third Order Shear Deformation Theory (TSDT): c_0=0, c_1=1, c_2=0, c_3=-(4/3)h

Suresh et al. [16] investigated the effect of warping on the free vibration of torsional-flexural coupled beams. Abramovich [17] studied free vibration of symmetrically laminated composite beams with the term representing the joint action of shear deformation and rotary inertia that was omitted in the Timoshenko equations. Song and Librescu investigated the formulation of the dynamic problem of laminated composite thick- and thin-walled, single-cell closed [18] and open [19] beams of arbitrary cross-section and their associated free vibration behavior. Banerjee [20, 21] proposed exact expressions for the frequency equation and mode shapes of symmetric beams with cantilever end conditions. He took into account the effect of material coupling between the bending and torsional modes of deformation together with the effects of shear deformation and rotary inertia. Cortinez and Piovan [22] developed a theoretical model for the dynamic analysis of composite thin-walled beams with open or closed cross-sections. Their model incorporated the shear flexibility as well as a state of initial stresses. Lee et al. [23] derived differential equations governing the free vibrations of elastic, horizontally curved beams with unsymmetric axes in cartesian coordinates, including the effect of torsional inertia. They computed numerically the frequencies and mode shapes for parabolic curved beams with both clamped ends and both hinged ends. In another research [24] they included the effects of axial extension, shear deformation, and rotatory inertia. Karama et al. [25] proposed a multi-layer laminated composite structure model to predict the mechanical behavior of multi-layered laminated composite structures using exponential function as a shear stress function. They validate the model for different cases in bending, buckling and free vibration on a cross-ply laminate. Kim et al. [26] studied the effects of the steel core or casing on the bending natural frequency of composite shafts for simply supported boundary conditions.

Li et al. [27] analyzed the vibration of axially loaded symmetrically laminated thin-walled beams with bending–torsion coupling by using the general solution of the equations of motion based on Timoshenko beam theory. Mei [28] presented the effect of coupling between bending and torsional deformations on vibrations of symmetric composite Timoshenko beams from a wave vibration standpoint. He found that the torsional modes at low frequencies and cutoff transitional frequency are unaffected by material coupling. Mei [29] also presented wave vibration analysis of axially loaded bending-torsion coupled composite Timoshenko beam structures.

First Order Shear Deformation Theories

First order Shear deformation theories (FSDT) were used by many authors. The work by Chandrashekhara et al. [30], Krishnaswamy et al. [31], Abramovich et al. [32, 33] was validated for symmetric cross-ply laminates that have no coupling. In the FSDT model by Teboub and Hajela [34] symmetric beams having fibers in one direction were considered. Bert and Kim [35] proposed a FSDT for predicting the critical speed of a shear deformable, composite driveshaft. They modeled the shaft as a Bresse-Timoshenko beam (FSDT with rotary inertia and gyroscopic action) generalized to include tending-twisting coupling. The FSDT models by Eisenberger et al. [36] and Qatu [37] for curved beams were also validated for cross-ply laminates.

Hajianmaleki and Qatu [38-39] considered different stiffness parameters and showed that using equivalent modulus of elasticity of each lamina (eq. 5) for calculation of ABD parameters (eqs 6-9) one can reach a model capable of static and dynamic analysis of generally laminated thick beams and shafts using FSDT.

$$\frac{1}{E_x^k} = \frac{\cos^4(\theta_k)}{E_{11}} + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_{11}}\right)\cos^2(\theta_k)\sin^2(\theta_k) + \frac{\sin^4(\theta_k)}{E_{22}} \tag{5}$$

$$A_{11} = \sum_{k=1}^{N} bE_x^k \left(h_k - h_{k-1}\right) \tag{6}$$

$$B_{11} = \sum_{k=1}^{N} bE_x^k \frac{\left(h_k^2 - h_{k-1}^2\right)}{2} \tag{7}$$

$$D_{11} = \sum_{k=1}^{N} bE_x^k \frac{\left(h_k^3 - h_{k-1}^3\right)}{3} \tag{8}$$

Higher Order Shear Deformation Theories

Some researchers developed and used higher order shear deformation theories (HSDT) for analysis of composite beams. Carrera and Petrolo [40] worked on the effectiveness of higher-order terms in refined beam theories. They concluded that the kinematics model that suits specific problem is determined by the cross-section geometry and the loading case. Khdier and Reddy [41] determined natural frequencies of the third-order, second-order, first-order and classical arch theories for cross-ply laminates. In another research they developed analytical solutions of refined beam theories to study the free vibration behavior of cross-ply rectangular beams with arbitrary boundary conditions in conjunction with the state space approach. They showed that the disagreement between different shear deformation theories is much less than the disagreement between any of them and EB theory [42].

Suresh and Nagaraj [43] proposed a HSDT for the static and dynamic analysis of thin-walled composite beams of arbitrary lay-ups and cross sections. Their method was applicable to beams of open as well as closed cross sections and was validated by comparison with experimental and analytical results for static deflections of composite beams with symmetric and antisymmetric lay-ups. Song and Waas [44] studied buckling and free vibration of stepped laminated composite beams using HSDT assuming a cubic distribution for the displacement field through the thickness. The results were compared to CBT and FSDT and did not show significant differences to those from Timoshenko theory for a wide range of aspect ratios of the beam geometry and material properties. Marur and Kant [45] proposed three higher order refined displacement models for the free vibration analysis of sandwich and composite beams. All higher order models were found to compute frequencies which were numerically higher than those of FSDT for the thin beams considered. In the case of thick sandwiches, higher order theories give quite significantly lower frequencies than Timoshenko theory. Kant et al. [46] proposed an analytical higher order model using C^0 continuity functions for symmetric laminates. Matsunaga [47] studied vibration and buckling of cross-ply laminated beams according to HSDT. Subramanian [48] proposed two higher order and two finite element approaches and validated them for symmetric laminated beams and different slenderness ratios.

Machado et al. [49] investigated dynamic stability of thin-walled composite beams, considering shear deformation, subjected to axial external force. They used Galerkin method in order to discretize the governing equation and the Bolotin’s method to determine the regions of dynamic instability of a simply supported beam. The numerical results showed that longitudinal vibration has large influence when the forcing frequency approaches the natural longitudinal frequency, obtaining parametric instability regions substantially wider. Machado and Cortinez [50] showed that when the ratio of the smaller axis flexural stiffness to the major axis flexural stiffness is large, classic analysis of vibration may lead to inaccurate predictions because of the effects of initial displacements.

Zhen and Wanji [51] assessed and compared different displacement-based theories for vibration of special symmetric and anti-symmetric composite and sandwich beams including zigzag model and different higher order theories. They proposed the global-local higher order theory to be suitable for symmetric and anti-symmetric laminates. Emam and Nayfeh [52] proposed a closed-form solution for the postbuckling deformation as a function of the applied axial load, which is beyond the critical buckling load. They exactly solved the linear vibration problem around the first buckled configuration. El Fatmi and Ghazouani [53] proposed a HSDT that can be viewed as an extension of Saint-Venant’s theory. Based on a kinematics built from the exact form of Saint-Venant displacement, their theory was rigorously derived for the case of symmetric cross-section made of orthotropic materials.

Layerwise Theories

In general, layerwise theories are used to represent local effects more accurately, such as interlaminar stress distribution, delaminations, etc. These theories are typically employed for cases involving anisotropic materials in which transverse shear effects cannot be ignored. Singh and Gupta [54] presented formulations for the rotor dynamic analysis of a composite rotor by using conventional equivalent modulus beam theory (EMBT) and a layerwise beam theory. They used Rayleigh-Ritz displacements for deriving the solution. The results indicated that the difference between the two theories was not large, but for unsymmetric stacking sequences with bending stretching couplings present the EMBT may result in inaccurate predictions of rotordynamic behavior. Lee [55] used a layerwise theory for free vibration analysis of a laminated beam with delaminations.

Other Theories

Pai and Nayfeh [56] proposed nonlinear equations describing the extensional-flexural-flexural-torsional vibrations of slewing or rotating metallic and composite beams. Three consecutive Euler angles were used to relate the deformed and undeformed states. The nonlinear equations of motion were used to investigate the response of an inextensional, symmetric angle-ply beam to a harmonic base-excitation along the flapwise direction [57] and forced nonlinear vibration of a symmetrically laminated beam

[58]. Kapuria et al. [59] used zigzag theory to satisfy continuity of transverse shear stress through the laminate to assess the dynamic and buckling response of laminated beams. Kovacs [60] proposed an iterative laminate model that can accurately determine the dynamic stress distribution in soft and hard cored sandwich arches.

Lee [61] applied pseudospectral method to the free vibration analysis of circularly curved multi-span Timoshenko beams and computed natural frequencies in good agreement with the literature. Sapountzakis and Dourakopoulos [62, 63] developed a boundary element method (BEM) for the general flexural-torsional vibration problem of Timoshenko beams of arbitrarily shaped composite cross-section taking into account the effects of warping stiffness, warping and rotary inertia and shear deformation. Six boundary value problems were solved using the Analog Equation Method, a BEM-based method.

Methods for Solving Equations of Motion

Differential Transform Method

The differential transform method (DTM) is a semi-analytic transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. In this method, certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem. The basic definitions and the application procedure of this method can be introduced as follows:

Consider a function f(x) which is analytic in a domain D and let x = x0 represent any point in D. The function f(x) is then represented by a power series whose center is located at x_0. The differential transform of the function f(x) is given by

$$F[k]=\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{9}$$

where f(x) is the original function and F[k] is the transformed function.

The inverse transformation is defined as

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F[k] \tag{10}$$

Combining Eqs. (9) and (10) gives

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{11}$$

In actual applications, the function f(x) is expressed by a finite series and Eq. (11) can be written as follows:

$$f(x)=\sum_{k=0}^{m}\frac{(x-x_0)}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \tag{12}$$

which means that $f(x)=\sum_{k=m+1}^{\infty}\frac{(x-x_0)}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}$ is negligibly small. The value of m depends on the convergence rate of the natural frequencies. By applying this transform to equations of motion, one can get solution to different problems and study the vibration of laminated beams. Ozgumus and Kaya [64] introduced the DTM to study the vibration characteristics of a rotating tapered cantilever EB beam with linearly varying rectangular cross-section of area proportional to x^n . In other researches they studied the out-of-plane free vibration analysis of a double tapered EB beam, mounted on the periphery of a rotating rigid hub [65] and performed free vibration analysis of a rotating, double tapered Timoshenko beam featuring coupling between flapwise bending and torsional vibrations using DTM [66]. Kaya and Ozgumus [67, 68] analyzed the free vibration response of an axially loaded closed-section composite Timoshenko beam which featured material coupling between flapwise bending and torsional vibrations using DTM.

Other people who used DTM were Catal [69] who solved differential equations of motion for free vibration of axially loaded and supported on elastic soil beam and Arikoglu and Ozkol [70] who made vibration analysis of composite sandwich beams with viscoelastic core.

Dynamic Stiffness Matrix Method

Abramovich et al. [71] proposed the exact element method to find vibration frequencies of multi-span laminated beams, including the effect of rotary inertia and shear deformation. They derived the dynamic stiffness matrix to solve for any set of boundary conditions including elastic connections, number and length of spans, as in the classical direct stiffness method for framed structures. They validated their model for symmetric cross-ply laminates. Banerjee and Williams [72, 73] used dynamic stiffness matrix in conjunction with the Wittrick-Williams algorithm to compute the natural frequencies and mode shapes of composite beams with substantial coupling between bending and torsional displacements. Banerjee [74] studied free vibration analysis of axially loaded composite Timoshenko beams by using the dynamic stiffness matrix method. Banerjee and Su [75] developed the dynamic stiffness matrix of a thin-walled spinning composite beam and investigated its free vibration characteristics based on CBT. Tseng et al. and Huang et al. [76, 77] developed an analytical solution for the free vibration of composite laminated beams of variable curvature using Timoshenko-type curved beam theory and incorporating the dynamic stiffness method.

Tseng et al. [78] combined the dynamic stiffness method with the Laplace transform to obtain accurate transient responses of an arch with variable curvature. They considered the effects of shear deformation, rotary inertia, and damping. Huang et al. [79] developed dynamic stiffness matrix for noncircular curved beams with variable cross-section to derive an exact solution of the out-of-plane free vibration. Jun et al. [80] used dynamic stiffness approach to numerically investigate vibration of laminated composite beams with arbitrary ply orientation. They accounted for the influences of Poisson effect, shear deformation and rotary inertia. They included the effect of axial load in another research [81]. Moon-Young et al. [82] proposed the exact dynamic and static stiffness matrices of shear deformable non-symmetric thin-walled beam-columns. They found the total potential energy in the general form by introducing the displacement field based on semitangential rotations and deriving transformation equations between displacement and force parameters defined at the arbitrary axis and the centroid-shear center axis, respectively.

State Space Approach (Transfer Matrix Method)

The state space approach (transfer matrix method) in vibration analysis of beams was first used by Khdeir and Reddy [83]. They studied free vibration of cross-ply laminated beams with arbitrary boundary conditions and presented the fundamental natural frequencies based on the various beam theories. Yildrim et al. [84] studied the in-plane free vibration problem of symmetric cross-ply laminated beams based on the transfer matrix method. They compared the results of Timoshenko and EB theories for first six non-dimensional frequencies with each other for length-to-thickness ratios. They included out of plane vibration in another research [85]. Yildrim [86] studied axial and shear deformation effects on the in-plane natural frequencies of symmetric cross-ply laminated circular arches and obtained the exact in-plane element stiffness matrix based on the transfer matrix method. In another research the effect of the longitudinal to transverse moduli ratio on the first three in-plane natural frequencies for different length/thickness ratios and boundary conditions of symmetric cross-ply beams was investigated [87]. Yildrim and Kiral [88] studied the effect of shear deformation and rotary inertia on out-of-plane free vibration problem of symmetric cross-ply laminated beams by the transfer matrix method. Yildrim [89] also did a numerical study to investigate the common effects of the rotary inertia and shear deformation on the first six out-of-plane free vibration frequencies of symmetric cross-ply laminated bars with the help of transfer matrix method.

Finite Element Methods

Finite element methods (FEM) are used in solving equations of motion mainly for incorporation of nonlinear effects and complicated geometries. Kosmatka and Friedmann [90] determined the free vibration characteristics of composite turbo-propellers using a number of straight beam-type finite elements. The FEM was obtained from Hamilton's principle, with allowances for generally anisotropic material behavior, arbitrary cross-sectional properties, large pretwist angles, out-of-plane cross-section warping, and geometrically nonlinear behavior, based on moderate-deflection theory. Singh et al. [91] investigated large amplitude free vibrations of unsymmetrically laminated beams using von Karman large deflection theory. They studied the problem by reducing the dynamic nonlinear finite element equations to two second order ordinary nonlinear differential equations using converged normalized spatial deformations in the positive and negative deflection half-cycles.

Hodges et al. [92] compared different methods for determination of cross sectional stiffness parameters and solved the equations of motion for eigenvalues using numerical integration and mixed FEM. Both of these methods were validated for symmetric beams. Lee et al. [93] proposed a FEM for stress and vibration analysis of laminated composite beams based on a multilayered theory. Their theory accounted for the continuity of interlaminar shear stress. Chandrashekhara and Bangera [94] developed a FEM based on HSDT to study the free vibration characteristics of laminated composite beams. They incorporated Poisson effect and rotary inertia and verified the numerical results for symmetrically laminated beams.

Chen et al. [95] derived finite element stiffness and consistent mass matrices for helically wound, symmetrical composite tubes. They used shell theory and lamination theory to formulate element stiffness matrices and reduced it to symmetrically laminated composite beam. Nabi and Ganesan [96] developed a general finite element based on a FSDT to study the free vibration characteristics of laminated composite beams. Their formulation accounted for bi-axial bending as well as torsion. Jeon et al. [97] investigated static and dynamic behavior of composite box beams using a large deflection beam theory. They obtained finite element equations of motion for beams undergoing arbitrary large displacements and rotations, but small strains, from Hamilton's principle. Gadelrab [98] used the FEM to obtain the effect of the delamination length and its starting point from the end condition on the natural frequencies of composite laminated beams. Rao and Ganesan [99] investigated the harmonic response of tapered composite beams using FEM. They incorporated effects of in-plane and rotary inertia as well as the Poisson effect and considered uniaxial bending only. Winfield et al. [100] used FEM to study the free vibration of a long thick laminated conical tube with a beam-type model. Zeng [101] developed the beam element of the composite element method. He presented the detailed numerical verifications for the beam element of Composite Element Method which involved the h-version and the c-version. Patel et al. [102] studied nonlinear free flexural vibrations and post-buckling of laminated orthotropic thick beams resting on a class of two parameter elastic foundation using a three noded shear flexible beam element. Geometric nonlinearity was considered using von Karman strain displacement relations. They solved nonlinear governing equations for orthotropic and cross-ply laminated beams with simply supported boundary conditions by employing the direct iteration technique.

Bassiouni et al. [103] used FEM in order to obtain the natural frequencies and mode shapes of laminated composite beams. They included shear deformation but not interfacial slip or delamination and compared numerical results with the experimental ones. The theoretical model gave good results compared with the experimental ones. Shi and Lam [104] presented a finite element formulation for the free vibration analysis of composite beams based on the third order beam theory. They studied the influence of the mass components resulting from higher order displacements on the frequencies of flexural vibration. Zapfe and Lesieutre [105] proposed a discrete layer beam finite element for the dynamic analysis of composite sandwich beams with integral damping layers. Raveendranath et al. [106] proposed a 2-noded curved composite beam element with three DOF per node for the analysis of laminated beam structures. The formulation accounted for flexural, extensional and transverse shear loadings in the plane of the curved beam based on FSDT. Their test problems prove the versatility of the element for the analysis of curved and straight laminated beams.

Yanchu [107] decomposed the laminate as multiple basic layers and assembled mass and stiffness matrices of these basic layers together to generate those matrices for the beam element. His method allowed the direct consideration of complex modulus of any layer so that it offered much more accurate damping analysis for such structures. Kapania and Goyal [108] developed three models to predict randomness in the free vibration response of unsymmetrically laminated beams: exact Monte Carlo simulation, sensitivity-based Monte Carlo simulation, and probabilistic FEA. Results showed that variations of 5 deg in ply angles have little effect on the lower mode natural frequencies of unsymmetrically and symmetrically laminated beams. Lee and Kim [109, 110] developed a general analytical model applicable to the dynamic behavior of a thin-walled channel section composite beams based on the CBT that accounts for the coupling of flexural and torsional modes for arbitrary laminate stacking sequence configuration. Piovan and Cortnez [111] carried out parametric studies of the natural frequencies of tailored composite thin-walled curved box-beams by means of FEM. Their structural model took into account the shear flexibility due to warping as well as due to bending.

Kim et al. [112] proposed a formulation for free vibration and spatial stability of non-symmetric thin-walled curved beams considering variable curvature effects and the second-order terms of finite semitangential rotations. They developed a thin-walled curved beam element using the third-order Hermitian polynomials and compared the numerical solutions with the results analyzed by ABAQUS shell elements. Ostachowicz and Zak [113] used HSDT based finite elements to study damped vibration of a laminated cantilever beam with a single closing delamination. Mitra et al. [114] developed a new composite thin wall beam element of arbitrary cross-section with open or closed contour. Their formulation incorporated the effect of elastic coupling, restrained warping, transverse shear deformation associated with thin walled composite structures. A FSDT approach was considered for static and free vibration analyses. Murthy et al. [115] derived a refined 2-node, 4 DOF/node beam element is based on higher order shear deformation theory for axial–flexural-shear coupled deformation in asymmetrically stacked laminated composite beams. The numerical results were validated for static and free vibration analysis of cross-ply beams.

Sarikanat et al. [116] determined the effects of axial load on the natural frequency in simply supported thick composite beams using FEM. The material properties of the elements were calculated with two different average value methods (arithmetic and weighted average). They observed that the results obtained by the arithmetic average method were quite close to the analytic results. Saravanos et al. [117] developed a 3D shear beam finite element for the damping analysis of tubular laminated composite beams. Sapountzakis and Mokos [118] developed a boundary element method to perform dynamic analysis of 3-D composite beam elements restrained at their edges by the most general boundary conditions and subjected in arbitrarily distributed dynamic loading. Piovan and Cortinez developed a theoretical model for the generalized linear analysis of composite thin-walled straight [119] and curved [120] beams with open or closed cross-sections incorporating full form of the shear deformability. They used the theoretical formulation together with a non-locking fourteen DOF finite element for the solutions to the general equations of thin-walled shear deformable composite beams. Ganesan and Zabihollah [121, 122] investigated the free undamped vibration response of tapered composite beams, using a higher-order finite element formulation. They determined stiffness coefficients of the tapered laminated beam based on the stress and strain transformations and classical laminate.

Duan [123] presented a finite element formulation for the nonlinear free vibration of thin-walled curved beams with non-symmetric open cross section. Jun et al. [124] proposed a dynamic FEM for free vibration analysis of generally laminated composite beams on the basis of FSDT by incorporating influences of Poisson effect, couplings among extensional, bending and torsional deformations, in the formulation. Boukhalfa et al. [125] employed a p-version, hierarchical finite element to investigate dynamic behavior of the rotating composite shaft on rigid bearings. They incorporated the transverse shear deformation, rotary inertia and gyroscopic effects, as well as the coupling effect. Kadioglu and Iyidogan [126] investigated free vibration of laminated composite curved beams using mixed FEM. Vo and Lee [127] developed a general analytical model based on SDBT to study the flexural–torsional coupled vibration and buckling of thin-walled open section symmetric laminated beams with arbitrary lay-ups using FEM. They studied effect of constant axial force using CBT [128] and SDBT [129].

In another research they developed a displacement-based 1D FEM with seven DOFs per node based on CBT to present the interaction curves for vibration and buckling of thin-walled composite box beams under constant axial loads and equal end moments [130]. Kim and Wang [131] carried out vibration analysis of composite beams by using FEM-based formal asymptotic expansion method. They used 3D equilibrium equations in which cross-sectional coordinates were scaled by the characteristic length of the beam and discretized microscopic 2D and macroscopic 1D equations obtained via the asymptotic expansion by applying a conventional FEM.

Experimental Investigation

Cudney and Inman [132] derived a method of estimating the distributed damping parameters of a beam based on frequency and damping ratios. Three different mathematical models were used to model the damping mechanism of a quasi-isotropic pultruded cantilevered beam. These three models were viscous damping, strain-rate damping, and both viscous and strain-rate damping. It was found by experimental modal analysis that the two-parameter damping model provides the best fit to measure modal data. Chandra and Chopra [133] presented a theoretical-cum-experimental study of the free vibration characteristics of thin-walled rotating box beams with bending-twist and extension-twist coupling using Galerkin method. The experimental frequencies and mode shapes correlated satisfactorily with the theoretical results. It was shown that bending-shear coupling influences the flexural vibration frequencies of antisymmetric box beams significantly while Extension-shear coupling does not. Bassiouni et al. [103] used a finite element model in order to obtain the natural frequencies and mode shapes of laminated composite beams. They included shear deformation but not interfacial slip or delamination and compared numerical results with the experimental ones. The theoretical model gave good results compared with the experimental ones. Baba and Thoppul [134] made a systematic experimental study to determine the effect of curvatures and debond on the flexural stiffness and strength of composite sandwich beam structures. Ooijevaar et al. [135] used experiments to investigate vibration based damage identification method for a composite T-beam.

Smart Beams

Composite materials have the advantage of easily placing smart sensors and actuators inside the structure. Smart materials such as piezoelectric sensors and actuators, shape memory alloys, and electrorheological fluids are used in studies of beams mainly for control of vibration. Choi et al. [136] studied the effect of electrorheological fluid on the vibration characteristics of a composite beam. They obtained the complex moduli of a hollow beam filled with an electrorheological fluid by analyzing the beam's motion in free oscillation.

Piezoelectric Beams

Kim and Jones [137] analytically investigated a quasi-static control strategy using embedded piezoelectric actuators to alter the vibration response of a composite beam model. Nonlinear equations of motion of the composite beam were developed, which incorporated the influence of initial in-plane loads generated by the piezoelectric layers. Spearritt and Asokanthan [138] proposed a theoretical and experimental development of a laminated spatially distributed piezoelectric torsional vibration actuator for a clamped-free cantilever beam. Their theoretical and experimental decay histories of the first torsion mode indicated a reduction in decay time for the controlled beam of greater than 10 times that of the uncontrolled beam.

Chandrashekhara and Varadarajan [139] worked on adaptive shape control of beams with piezoelectric actuators. Takawa et al. [140] studied flexural–torsion coupled vibration control of a composite cantilevered beam by using piezoceramic actuators in two different directions. Peng et al. [141] developed a FEM based on third order laminate theory for the active position and vibration control of composite beams with distributed piezoelectric sensors and actuators. Shih [142] studied the distributed vibration sensing and control of a piezoelectric laminated curved beam. Chattopadhyay et al. [143] investigated vibration reduction in rotor blades using active composite box beam. They used a 3D model that approximated the elasticity solution so that the beam cross-sectional properties were not reduced to 1D parameters and both in-plane and out-of-plane warpings were included.

Varadarajan et al. [144] developed a FEM based on HSDT, accounting for piezoelectric effects to investigate performance of an LQG/LTR-based multi-input multi-output robust vibration control system for a laminated composite beam. Takawa et al. [145] investigated the fuzzy control of vibration for a hybrid smart CFRP cantilevered beam actuated by piezoceramic and electro-rheological fluids actuators. Abramovich and Livshits [146] developed a balanced model describing the behavior of piezolaminated composite beams based on FSDT. Raja et al. [147] used quasi-static equations of piezoelectricity to derive a FEM capable of modeling two different kinds of piezoelastically induced actuation in a sandwich beam. They developed a control scheme based on the linear quadratic regulator/independent modal space control method and estimated the active stiffness and the active damping introduced by shear and extension-bending actuators. The shear actuator was more efficient in controlling the vibration than the extension-bending actuator for the same control effort.

Waisman and Abramovich [148] investigated the stiffening effects of a simply supported and clamped–free symmetric piezolaminated composite type beam. Song et al. [149] studied the active vibration control of a cantilever composite beam using piezoceramic patches using both theoretical and FEM analysis. Takawa and Fukuda [150] and Susumu and Takeshi [151] used a fuzzy model to investigate vibration control for a smart CFRP beam actuated by piezoceramic and electrorheological fluid actuators.

Mitra et al. [152] made theoretical and experimental investigation of vibration control of composite box beams using distributed, surface mounted piezoelectric actuators. The finite element modeling of their box beam was done by formulating a FSDT active composite thin walled beam element. Chandiramani et al. [153] worked on optimal vibration control of a rotating composite beam with distributed piezoelectric sensing and actuation. They modeled a rotating composite blade, as a box-beam with transverse shear flexibility, shear-tractionless bounding faces and restrained warping, and subjected to a time dependent pressure pulse. Librescu and Na [154] studied the vibration control of adaptive doubly-tapered cantilevered composite beams, simulating an aircraft wing, exposed to time-dependent external pulses through the converse piezoelectric effect.

Gu and Song [155] used piezoceramic patch sensors and actuators for active vibration suppression of a composite I-beam using fuzzy positive position control. Lin and Nien [156] investigated modeling and vibration control of a smart beam using piezoelectric damping-modal actuators/sensors. Sethi et al. [157] and Sethi and Song [158] studied and tested vibration control of a flexible composite I-beam using piezoceramic sensors and actuators. Edery-Azulay and Abramovich [159] studied the effects of piezoceramic materials on the augmented damping of vibrating piezo-composite beams. Ashida et al. [160] investigated control of thermally induced vibration in a composite beam with damping effect. Their beam consisted of a central thermoelastic structural layer and two outer piezothermoelastic layers.

Choi et al. [161] studied bending vibration control of the pre-twisted rotating composite thin-walled beam. The formulation was based on single cell composite beam including a warping function, centrifugal force, Coriolis acceleration, pre-twist angle and piezoelectric effect. Fridman and Abramovich [162] used FSDT to compute natural frequencies and their associated mode shapes, as well as, buckling loads for beams with and without piezoelectric layers influence, having various boundary conditions and lay-ups.

Ji et al. [163] proposed an improved semi-active control synchronized switch damping on voltage (SSDV) method and applied to the vibration control of a composite beam. Also Ji et al. [164] proposed an adaptive semi-active SSDV method based on the least mean squares (LMS) algorithm and applied to the vibration control of a composite beam. Susanto [165] presented an analytical model of piezoelectric laminated slightly curved beams, which included the computation of natural frequencies, mode shapes and transfer function formulation using the distributed transfer function method (DTFM).

Vadiraja and Sahasrabudhe [166] used HSDT for Structural modeling of rotating pre-twisted thin-walled composite beams with embedded macro fiber composite actuators and sensors. It was observed that gyroscopic coupling between lagging-extension motions had significant effect and cannot be neglected in the analysis. Foda et al. [167] developed an analytical approach to suppress the steady state transverse vibration of a symmetric cross-ply laminated composite beam that is excited by an external harmonic force by piezoelectric patches. They used dynamic Green’s functions to solve governing equations and proposed a scheme for determining the values of the driving voltages, the dimensions of the PZT patches and their locations along the beam. Chandiramani [168] studied the optimal control of a thin-walled rotating beam using HSDT. The pretwisted, doubly tapered beam was comprised of orthotropic host with surface-embedded transversely isotropic piezoelectric sensor-actuator pairs. Abramovich [169] showed that in-plane cross sectional deformations due to small lateral vibrations, when coupled with the internal stresses due to the constant electric voltage, results in a significant influence on the free lateral vibrations.

Beams with Shape Memory Alloys

Lau et al. [170] investigated the change of natural frequencies of a clamped–clamped composite beam with embedded shape memory alloy wires. Tsai and Chen [171] investigated numerical parametric studies of the natural frequencies and static buckling loads of the composite beam with activated SMA fibers.

Aoki and Shimamoto [172] investigated the active damping effect of the smart matrix composite made of epoxy resin with embedded fibers of TiNi to examine if it is possible to apply the composite as a damping material. Zhang and Zhao [173] studied the kinematic assumptions influence on deflection and vibration characteristics of a composite beam with arbitrarily embedded shape memory alloy. They compared natural frequencies of the composite beam with the nonlinear governing equation, which were obtained by directly linearizing the equations and locally linearizing the equations around each equilibrium. Majewska et al. [174] investigated active vibration control of a cracked composite beam using magnetic shape memory alloy actuators. Lee et al. [175] used transfer matrix method to study lateral vibration of a composite stepped beam consisted of SMA helical spring based on CBT. A discussion on this paper was made by Sinha [176]. Dos Reis et al. [177] studied vibration attenuation in an epoxy smart composite beam with embedded NiTi shape memory wires.

Complicating Effects

Dynamic Loading and Excitation

Gong and Lam [178] studied transient response of layered composite beams subjected to underwater shock. Ganesan and Kowda [179] investigated free vibration of composite beam-columns with stochastic material and geometric properties subjected to random axial loads. They used the perturbation method in the context of stochastic analysis.

Jun and Xianding [180] studied the flexure–torsion coupled random response of composite beams with solid or thin-walled closed sections subjected to various types of concentrated and distributed random excitations. They assumed random excitations to be stationary, ergodic and Gaussian and obtained analytical expressions for the displacement response of the composite beams by using normal mode superposition method combined with frequency response function method.

Li et al. [181] investigated the stochastic bending–torsion coupled response of axially loaded slender composite beams with solid or thin-walled closed cross sections by using normal mode method in conjunction with receptance method. They used CBT with the effects of bending–torsion coupling and axial force included. Kiral [182] used a 3D FEM based on the classical laminated plate theory together with the Newmark integration method in order to obtain the dynamic response of the beam. He employed Rayleigh damping in the dynamic analyses and presented the impulse, step and moving load responses of the composite beam are for different damping ratios. Ibrahim et al. [183] investigated the periodic response of cross-ply composite curved beams subjected to harmonic excitation with frequency in the neighborhood of symmetric and antisymmetric linear free vibration modes. Their analysis was carried out using HSDT based FEM.

Rotating Beams

Rotating beams are mainly used in two applications. First, shafts that have tubular cross section and spin about the longitudinal axis and second, blades that have box cross section and rotate along normal axis. The first is mainly used in the automotive industry and power transmission devices while the second have mainly aerospace applications.

Shafts

Singh et al. [184] reviewed the developments in dynamics of composite material shafts in 1997. Singh and Gupta [185] investigated on natural frequencies and damping ratios in flexural modes of composite cylindrical tubes. They used beam and shell formulation and concluded that their beam theory can not account for bending-stretching and shear-normal couplings. Also for extreme values of length and thickness to radius ratio, the beam formulation was inaccurate. Kim and Bert [186] proposed a theoretical analysis for determining the critical speeds of a rotating circular cylindrical hollow shaft by means of the thin-and thick-shell theories. They used the dynamic analog of the Sanders best first approximation shell theory. They included the combined effects of torsion and rotational effects containing the centrifugal and Coriolis forces. Song and Librescu [187] investigated on anisotropy and structural coupling on vibration and instability of spinning thin-walled beams. They included transverse shear and the primary and secondary warping effects. Kim et al. [188] Investigated on free vibration of a rotating tapered composite Timoshenko shaft using Galerkin method. It was found that by tapering, bending natural frequencies and stiffness can be significantly increased over those of uniform shafts having the same volume and made of the same material.

Song et al. [189] investigated on vibration and stability of anisotropic pretwisted beams rotating at constant angular speed about the longitudinal body-axis. They used refined theory of thin-walled anisotropic composite beams featuring bending-bending elastic coupling. In a subsequent publication [190], they addressed problems related with the implications of conservative and gyroscopic forces on vibration and the stability of a circular cylindrical shaft modeled as a thin-walled spinning composite beam. Chang et al.

[191] considered composite shaft containing discrete isotropic rigid disks and supported by bearings modeled as springs and viscous dampers. They extended Hamilton's principle to derive the governing equations for finding critical speeds of composite shaft systems and incorporated the transverse shear deformation, rotary inertia and gyroscopic effects, as well as the coupling effect due to the lamination of composite layers. Chang et al. [192] performed vibration analysis of rotating composite shafts containing randomly oriented reinforcements. Gubran and Gupta [193] analyzed the natural frequencies of composite tubular shafts using equivalent modulus beam theory with shear deformation, rotary inertia and gyroscopic effects included. Their approach took into account effects of stacking sequence and different coupling mechanisms. Banerjee and Su [75] developed the dynamic stiffness matrix of a spinning thin-walled composite beam and investigated its free vibration characteristics based on CBT. Na et al. [194] studied vibration and stability of a circular cylindrical shaft modeled as a tapered thin-walled composite beam, spinning with constant speed and subjected to an axial compressive force. Ghoneim and Lawrie [195] developed a mathematical model, based on Timoshenko beam assumption, for a rotating composite cylindrical shaft with cylindrical constrained layer damping partially covering the length span of the shaft. Sino et al. [196] introduced a homogenized FEM which takes into account internal damping of the beam and evaluated natural frequencies and instability thresholds of an internally damped rotating composite shaft. Qatu and Iqbal [10] used CBT to solve the vibration of a cross-ply laminated composite driveshaft with an intermediate joint. Alwan et al. [197] studied dynamic behavior of composite shafts with particular interest on estimation of damping. They used ANSYS for modeling of the shaft and analyzed the effect of material properties and stacking

sequence on eigenfrequencies of composite tube-shafts. Different methodologies such as logarithmic decay curve, half-power method, and hysteresis loop method using force sensors were used for determining the damping of the composite shafts.

Blades

Kosmatka and Friedmann [90] determined the free vibration characteristics of composite turbopropellers using a number of straight beam-type finite elements. The FEM was obtained from Hamilton's principle, with allowances for generally anisotropic material behavior, arbitrary cross-sectional properties, large pretwist angles, out-of-plane cross-section warping, and geometrically nonlinear behavior, based on moderate-deflection theory. Chandra and Chopra [133] presented a theoretical-experimental study of the free vibration characteristics of thin-walled composite box beams with bending-twist and extension-twist coupling under rotating conditions using Galerkin method. The experimental frequencies and mode shapes correlated satisfactorily with the theoretical results. It was shown also that bending-shear coupling influences the flexural vibration frequencies of antisymmetric box beams significantly while extension-shear does not. Ozgumus and Kaya [65, 66] performed free vibration analysis of a rotating, double tapered beam featuring coupling between flapwise bending and torsional vibrations using DTM. Kuang and Hsu [198] investigated on effect of fiber angle, internal and external damping, inclined angle and the rotation speed on the natural frequencies of orthotropic composite pre-twisted blades by employing the differential quadrature method (DQM). Choi et al. [161] studied bending vibration control of the pre-twisted rotating composite thin-walled beam. The formulation was based on single cell composite beam including a warping function, centrifugal force, Coriolis acceleration, pre-twist angle and piezoelectric effect. They used CBT along Kelvin–Voigt internal and linear external damping coefficients. Huang et al. [199] proposed a method based on the power series solution to solve the natural frequency of very slender rotating beam at high angular velocity. They investigated the natural frequency of the flapwise bending vibration, and coupled lagwise bending and axial vibration for the rotating beam.

Damaged Beams

Study on damaged beams is focused on two main areas. One aspect of research is to investigate the effect of damages on beams natural frequencies in order to avoid those ranges. Another aspect is to use vibration for finding damages present in the structure that is called vibration monitoring. Vibration monitoring is one the main methods for damage identification and health monitoring of composite structures.

Damage Effect on Natural Frequencies

Della and Shu [200] provided a relevant survey on the various analytical models and numerical analyses for the free vibration of delaminated composites in 2007. Krawczuk and Ostachowicz [201] investigated modeling and vibration analysis of a cantilever composite beam with a transverse open crack. Gadelrab [98] used FEM to obtain the effect of the delamination length and its starting point from the end condition on the natural frequencies of composite laminated beams. Lee [54] used a layerwise theory for free vibration analysis of a laminated beam with delaminations. Lee et al. [202] made free vibration analysis of axially compressed laminated composite beam-columns with multiple delaminations. Birman and Byrd [203] investigated the effect of matrix cracks in longitudinal and transverse layers of cross-ply ceramic matrix composite beams on their mechanical properties and vibration frequencies. Kisa [204] used FEM and the component mode synthesis methods to perform free vibration analysis of a cantilever composite beam with multiple cracks. Ostachowicz and Zak [113] used HSDT based FEM to study damped vibration of a laminated cantilever beam with a single closing delamination. Perel [205] performed FSDT based FEM for vibration of delaminated composite beam with an account of contact of the delamination crack faces. Zak [206] studied damped non-linear vibration of a delaminated composite beam using HSDT based FEM. Della and Shu [207] presented an analytical solution for the free vibrations of beams with two overlapping delaminations in prebuckled states. They analyzed the delaminated beam as seven interconnected EB beams and observed a monotonic relation between the natural frequency and the compressive load. Baba and Gibson [208] used a 2D FEA to predict the natural frequencies and corresponding vibration modes of a free-free sandwich beam with delamination of various sizes and locations. Baba and Thoppul [134] made a systematic experimental study to determine the effect of curvatures and debond on the flexural stiffness and strength of composite sandwich beam structures. Free vibration analysis of delaminated composite beams was investigated by Kiral [209]. Yazdi and Rezaeepazhand [210] studied the applicability of similitude theory in establishing necessary similarity conditions for designing scaled down models for predicting the vibration behavior of delaminated composite beam-plates.

Vibration Monitoring

Zou et al. [211] conducted a review in 1998 on vibration based model independent damage identification and health monitoring of composite structures. Ratcliffe and Bagaria [212] presented an experimental nondestructive vibration-based technique for locating a delamination in a composite beam. Sahin and Shenoi [213] investigated the effectiveness of the combination of global (changes in natural frequencies) and local (curvature mode shapes) vibration-based analysis data as input for artificial neural networks for location and severity prediction of damage in composite beams.

Ooijevaar et al. [135] used experiments to investigate vibration based damage identification method for a 2.5-dimensional composite T-beam. Nichols and Murphy [214] worked on detecting delamination in composite beams based on a polyspectral analysis of the structure's vibrational response. They presented a low-dimensional model of the structure that captures the delamination-induced nonlinearity and showed how it influences the beam's dynamic response.

Added Mass Effect

Chandrashekhara and Bangera [215] investigated vibration of symmetrically laminated clamped-free beam with a mass at the free end. They derived equations of motion for the laminated beam accounting for the Poisson effect, rotary inertia and transverse shear deformation. White and Heppler [216] derived the equations of motion and boundary conditions for a free-free Timoshenko beam with rigid bodies attached at the endpoints and developed the frequency equation including the effects of the body mass, first moment of mass, and moment of inertia. Dadfarnia et al. [217] used a translational cantilevered EB beam with tip mass at its free end to study the effect of several damping mechanisms on the stabilization of the beam displacement. A Lyapunov-based controller utilizing a partial differential equation model of the translational beam was developed to exponentially stabilize the beam displacement while the beam support is regulated to a desired set-point position.

Damped and Viscoelastic Beams

Bishop and Kinra [218] studied thermoelastic damping of a laminated beam in flexure and extension. Zapfe and Lesieutre [219] proposed a smeared laminate model for the dynamic analysis of laminated beams to predict the modal frequencies and damping of simply supported beams with integral viscoelastic layers. Their model included the effects of transverse shear and rotary inertia. In another research they proposed a discrete layer beam FEM for the dynamic analysis of composite sandwich beams with integral damping layers [105]. Yim [220] compared three different methods for prediction of damping of a symmetric balanced laminated composite beam. Kovacs [221] proposed an iterative model to predict the modal frequencies and damping of simply-supported sandwich circular arch. He compared solutions for a three-layer circular arch with a three-layer approximate model.

Vengallatore [222] studied thermoelastic damping in symmetric, three-layered, laminated, micromechanical EB beams using an analytical framework developed by Bishop and Kinra. Numayr and Qablan [223] considered three cases to analyze free vibrations of wide sandwich beams. They modeled viscoelastic core by elastic translational and rotational springs and used the finite difference method to solve the problem. They showed that if the bending–torsion coupling was pronounced, the inclusion of warping affects the natural frequency considerably.

Saravanos et al. [117] developed 3D shear beam finite element for the damping analysis of tubular laminated composite beams. Edery-Azulay and Abramovich [159] studied the effects of piezoceramic materials on the augmented damping of vibrating piezo-composite beams. Prabhakar and Vengallatore [224] presented an exact theory to compute the frequency dependence of thermoelastic damping in asymmetric, bilayered, micromechanical EB beam resonators. Arvin et al. [225] made a numerical study of free and forced vibration of composite sandwich beam with viscoelastic core. They achieved higher order theory for sandwich beam with composite faces and viscoelastic core by considering independent transverse displacements on two faces and linear variations through the depth of the beam core. Arikoglu and Ozkol [70] made Vibration analysis of composite sandwich beams with viscoelastic core by using DTM. Alwan et al. [197] studied different methodologies such as logarithmic decay curve, half-power method, and hysteresis loop method using force sensors for determining damping of composite shafts.

Beams on Elastic Support

Patel et al. [102] studied nonlinear free flexural vibrations and post-buckling of laminated orthotropic beams resting on a class of two parameter elastic foundation using a three-noded shear flexible beam element. Their nonlinear formulation includes the effects of transverse shear deformation, in-plane and rotary inertia terms. Koutsawa and Daya [226] performed static and free vibration analysis of laminated fiberglass beam on viscoelastic supports. Jafari-Talookolaei and Ahmadian [227] investigated free vibration analysis of a cross-ply laminated composite beam on two parameter elastic foundation (Pasternak) foundation. They computed natural frequencies of beam on Pasternak foundation using FEM on the basis of Timoshenko beam theory. Malekzadeh and Vosoughi [228] did differential quadrature (DQ) large amplitude free vibration analysis of laminated composite thin beams on nonlinear elastic foundation. They considered beam edges to be elastically restrained against rotation and in-plane immovable. They developed a finite element program to verify the results of the presented DQ approach. Baghani et al. [229] presented analytical expressions for large amplitude free vibration and post-buckling analysis of unsymmetrically laminated composite beams on elastic foundation. They solved nonlinear governing equation by employing the variational iteration method and reached accurate solutions for a wide range of vibration amplitudes.

Other Complexities

Vibration analysis of stepped laminated composite Timoshenko beams was investigated by Farghaly and Gadelrab [230] and Dong et al. [231]. They determined flexural rigidity and transverse shearing rigidity of a laminated beam based on FSDT. Kütüg [232] investigated the frequency and forms of natural vibrations of a hinged beam-strip fabricated from a composite material with small-scale curved structures using the plate theory in the framework of the Timoshenko hypothesis.

Nachum and Altus [233] studied natural frequencies and mode shapes of non-homogeneous (deterministic and stochastic) rods and beams based on the functional perturbation method.

Concluding Remarks

Research on the analyses of composite beams has been increasing rapidly in the last two decades. Researchers in the field of composite beam analyses avoided use of 3D theory of elasticity and developed and used thick beam theories. More than 230 research articles have been cited on the subject during this time with more than two third in the last 10 years. Many researchers use the finite element method for their analyses. Much of the research is paying attention to evolving technologies on piezoelectric materials and applications like blades and shafts. Other research focused on vibration control through damping and structural health monitoring through vibration testing.

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CHAPTER III

STATIC AND VIBRATION ANALYSIS OF GENERALLY

LAMINATED STRAIGHT THICK BEAMS

Introduction

The use of laminated composite beams, plates and shells in many engineering applications including aerospace, marine, medical equipment, automotive as well as others has been expanding rapidly in the past decades. Structures composed of composite materials offer lower weight and higher strength and stiffness than those composed of most metallic materials. However, their main advantage is the possibility to tailor design for specific application. Composite beams, plates and/or shell components now constitute a large percentage of recent aerospace, submarine and other structures.

Besides, composites have their own challenges in predicting dynamic behavior, failure, or fatigue. The designers face more complicated problems in designing such structures and they should broaden their knowledge to efficiently employ these benefits while considering all phenomena related to composites. One of the important problems in composite structures design is the dynamic behavior and natural frequency calculation for composite beams. Composite beams are lightweight load carrying structures in many diverse applications including construction industries.

This chapter is concerned with the development of a general approach for the static and vibration analysis of generally laminated composite beams, which can be useful for design engineers. Three dimensional finite element analyses are based on the 3D elasticity theory. Thus, they are the most accurate analytical procedure to obtain natural frequencies. However, 3D FEM is a very expensive procedure demanding both expensive machines and longer computer times if used for large scale structures. In addition, these structures have two dimensions smaller than the third. For these structures, beam models are very efficient provided they are built on accurate models and are verified against 3D analyses as is done in this chapter. It should be mentioned here that the treatment presented in this chapter considers beams vibrating (or deforming) in one plane.

Two classes of theories are developed for laminated beams. In the first class of theories, thin beams are studied where effects of shear deformation and rotary inertia are neglected. This class of theories will be referred to as thin beam theories or classical beam theories (CBT). This is typically accurate for thin beams and fundamental frequencies and is less accurate for thicker beams and higher frequencies. In the second class of theories, shear deformation and rotary inertia effects are considered. This class of theories will be referred to as thick beam theory or shear deformation beam theory (SDBT).

The well-known formula for finding the natural frequencies of simply supported composite beams ($\omega_n = \left(n\pi / \ell\right)^2 \sqrt{D_{11} / \rho A}$) is based on assuming a thin beam theory and a symmetric cross-ply laminate. It cannot be used for thick laminates and those that have any kind of coupling.

The inclusion of shear deformation in the analysis of beams was first made by Timoshenko [1] and this problem has been incorporated by most researchers. But to the knowledge of the author, there is no simple approach for dynamic analysis of composite beams considering all kinds of couplings. Kapania, Raciti [2] conducted a review on advances in analysis of laminated beams and plates vibration and wave propagation in 1989. Chidamparam and Leissa [3] reviewed the published literature on the vibrations of curved bars, beams, rings and arches of arbitrary shape which lie in a plane in 1993. Among first order shear deformation theories (FSDT) the works by Chandrashekhara et al. [4], Krishnaswamy et al. [5], Abramovich et al [6, 7] were validated for symmetric cross-ply laminates that have no coupling. In the FSDT model by Teboub and Hajela [8] or SDT models by Banerjee [9, 10] or Lee at al. [11] symmetric beams having fibers in one direction (only bending-twisting coupling) were considered. The FSDT models by Eisenberger et al. [12] and Qatu [13, 14] for curved beams (that can be easily specialized to straight beams) was also validated for cross-ply laminates (only bending-stretching coupling).

Some researchers developed higher order shear deformation theories (HSDT) to address issues of cross sectional warping and transverse normal strains. Khdier and Reddy [15] determined natural frequencies of the third-order, second-order, first-order and classical arch theories for cross-ply laminates. Kant et al. [16] and Matsunaga [17] studied vibration of cross-ply laminated beams according to HSDT.

Subramanian [18] proposed two higher order and two finite element approaches and validated them for symmetric cross-ply laminated beams. Kapuria et al. [19] used zigzag theory to satisfy continuity of transverse shear stress through the laminate to assess the dynamic and buckling response of laminated beams. The results showed that the zigzag theory can be accurate for natural frequency calculations of beams with specific geometry and lay-up (symmetric or cross-ply laminates).

Zhen and Wanji [20] compared different displacement-based theories for vibration of composite and sandwich beams including Kapuria's model [19] and different higher order theories. They proposed the global-local higher order theory that was validated for cross-ply laminates.

In this chapter, classic and FSDT beam models will be evaluated for their accuracy in static and vibration analysis using different approaches for stiffness parameters calculation. Their results will be compared with those obtained using a 3D finite element model for different laminates (unidirectional, symmetric and asymmetric cross ply and symmetric and asymmetric angle-ply).

Composite Beams Stiffness Calculation

Figure 3.1 shows a free body diagram of a differential beam element. Beams are considered as one dimensional (1D) load carriers and the main parameter for analysis of load carrier structures is stiffness.

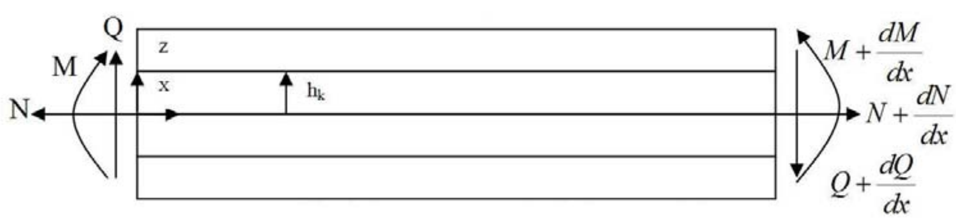

Figure 3.1 Free body diagram of a differential beam element

In general for composite laminates, the stiffness matrix composed of ABD parameters is used to relate the stress resultants to strains.

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{0x} \\ \varepsilon_{0y} \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{bmatrix} \tag{1}$$

where regular ABD stiffness parameters for beams are defined as.

$$A_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k}\left[\left(h_k - h_{k-1}\right)\right] \tag{2}$$

$$B_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k}\frac{\left(h_k^2 - h_{k-1}^2\right)}{2} \tag{3}$$

$$D_{11} = \sum_{k=1}^{N} b\overline{Q}_{11}^{k}\frac{\left(h_k^3 - h_{k-1}^3\right)}{3} \tag{4}$$

Note here that the above definitions are different from those used for general laminate analysis in the literature. The width in the above terms is included in the definitions of these terms, while it is customary to leave this term out in general laminate analysis. In 1D analysis of beams, as we will see later, only parameters in the x direction are considered and other parameters are ignored. So instead of 6X6 stiffness matrix for general laminate analysis we will have a 2X2 matrix for CBT and 3X3 matrix for SDBT. This formulation has the disadvantage of not accounting for any coupling other than the bending-stretching coupling terms B_{11}. To overcome this problem, we propose that instead of the normal definitions of A_{11}, B_{11}, and D_{11}, one can use equivalent stiffness parameters that include couplings. That is why we will deal with stiffness parameters first.

Kaw [24] proposed first finding the inverse of the ABD matrix (J matrix) for the whole laminate. He then defines the laminate modulus of elasticity as

$$E = \frac{b}{I\,J_{44}} \qquad J = [ABD]^{-1} \tag{5}$$

where J_{44} is the term in 4th row and 4th column of the inverse of the ABD matrix of the laminate and I is the moment of inertia. Then he defines D_{11} to be

$$\left(D_{11}\right)_k = \frac{b}{J_{44}} \tag{6}$$

In this chapter this approach is used as an option with assuming the following formula for A_{11} and B_{11}.

$$\left(A_{11}\right)_k = \frac{b}{J_{11}} \tag{7}$$

$$\left(B_{11}\right)_k = \frac{1}{J_{14}} \quad \text{(assumed zero for symmetric laminates)} \tag{8}$$

Rios and Chan [25] proposed the following formulation for parameters A_{11}, B_{11}, and D_{11}.

$$\left(A_{11}\right)_c = \frac{1}{a_{11} - \dfrac{b_{11}^2}{d_{11}}} \tag{9}$$

$$\left(B_{11}\right)_c = \frac{1}{b_{11} - \dfrac{a_{11}d_{11}}{b_{11}}} \tag{10}$$

$$\left(D_{11}\right)_c = \frac{1}{d_{11} - \dfrac{b_{11}^2}{a_{11}}} \tag{11}$$

where a_{11}, b_{11}, and d_{11} are relevant compliance matrix terms. According to the previous formulation we have $a_{11}=J_{11}$, $b_{11}=J_{14}$, $d_{11}=J_{44}$.

Method based on ply equivalent modulus

The formulation for the stiffness in equations 2-4 is based on plate formulation and two dimensional loading. However, one dimensional formulation is appropriate for beam formulation. If a uniaxial loading (σ_x) on an orthotropic lamina is assumed, where the fibers direction (direction 1) is at angle θ to reference axis x. The normal and shear stress through 1 and 2 axis is

$$\sigma_1 = \sigma_x \cos^2 \theta \tag{12}$$

$$\sigma_2 = \sigma_x \sin^2 \theta \tag{13}$$

$$\tau_{12} = -\sigma_x \sin\theta \cos\theta \tag{14}$$

The strains in the 1 and 2 directions are

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \upsilon_{21}\frac{\sigma_2}{E_2} = \sigma_x\left(\frac{\cos^2\theta}{E_1} - \upsilon_{21}\frac{\sin^2\theta}{E_2}\right) \tag{15}$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - \upsilon_{12}\frac{\sigma_1}{E_1} = \sigma_x\left(\frac{\sin^2\theta}{E_2} - \upsilon_{12}\frac{\cos^2\theta}{E_1}\right) \tag{16}$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} = -\frac{\sigma_x \sin\theta\cos\theta}{G_{12}} \tag{17}$$

The stains in the x and y directions can be found by the transformation law

$$\varepsilon_x = \varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta - \gamma_{12}\sin\theta\cos\theta \tag{18}$$

Substitution of equations 15-17 in equation 18 gives the strain

$$\varepsilon_x = \sigma_x \left[\frac{\cos^4\theta}{E_1} + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_1} \right) \cos^2\theta \sin^2\theta + \frac{\sin^4\theta}{E_2} \right] \quad (19)$$

Then the formulation for the equivalent modulus of elasticity of the lamina is

$$\frac{1}{E_x^k} = \frac{\cos^4(\theta_k)}{E_{11}} + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_{11}} \right) \cos^2(\theta_k)\sin^2(\theta_k) + \frac{\sin^4(\theta_k)}{E_{22}} \quad (20)$$

Figure 3.2 shows the variation of E_x and $\overline{Q_{11}}$ divided by E_1 as a function of laminate angle.

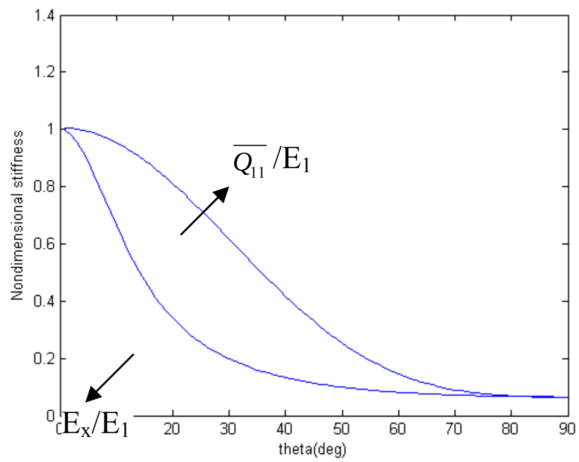

Figure 3.2 Variation of E_x and $\overline{Q_{11}}$ divided by E_1 as a function of laminate angle

Equivalent A_{11}, $B_{11}$and D_{11} using these formulas would be

$$(A_{11})_e = \sum_{k=1}^{N} bE_x^k (h_k - h_{k-1}) \quad (21)$$

$$(B_{11})_e = \sum_{k=1}^{N} bE_x^k \frac{(h_k^2 - h_{k-1}^2)}{2} \quad (22)$$

$$(D_{11})_e = \sum_{k=1}^{N} bE_x^k \frac{(h_k^3 - h_{k-1}^3)}{3} \quad (23)$$

Table 3.1 shows nondimensional A_{11}, B_{11}, and D_{11} for some laminates using different methods proposed.

Table 3.1 Nondimensional A_{11}, B_{11}, and D_{11} using different formulation for a graphite/epoxy beam with different laminates (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

Laminate	**A_{11}/E_2bh**			
	(A_{11})	**$(A_{11})_e$**	**$(A_{11})_f$**	**$(A_{11})_c$**
$[0]_4$	15.492	15.402	15.402	15.402
$[0/90]_s$	8.2491	8.2009	8.2381	8.2381
$[0_2/90_2]$	8.2491	8.2009	3.4681	8.2229
$[45]_4$	5.0678	1.7483	1.7483	1.7483
$[30_2/60_2]$	5.8632	2.1511	2.0592	2.6866
Laminate	**B_{11}/E_2bh^2**			
	(B_{11})	**$(B_{11})_e$**	**$(B_{11})_f$**	**$(B_{11})_c$**
$[0]_4$	0	0	0	0
$[0/90]_s$	0	0	0	0
$[0_2/90_2]$	1.8108	1.8002	1.3166	1.8051
$[45]_4$	0	0	0	0
$[30_2/60_2]$	0.9054	0.2236	1.6373	0.4988
Laminate	**D_{11}/E_2bh^3**			
	(D_{11})	**$(D_{11})_e$**	**$(D_{11})_f$**	**$(D_{11})_c$**
$[0]_4$	1.2910	1.2835	1.2835	1.2835
$[0/90]_s$	1.1401	1.1335	1.1374	1.1374
$[0_2/90_2]$	0.6874	0.6834	0.2890	0.6852
$[45]_4$	0.4223	0.1457	0.2716	0.2716
$[30_2/60_2]$	0.4886	0.1793	0.3040	0.3966

Static Analysis

In the static analysis section, we will consider composite beams loaded with classical loading conditions and derive differential equations for displacements. Those equations will be solved with classical boundary conditions of both ends simply supported, both ends clamped and cantilever boundary conditions. We will use the static analyses to find deflection and stress of composite beams under both CBT and SDBT using both Euler approach and matrix approach.

Classical Beam Theory

Applying the traditional assumptions for thin beams (normals to the beam midsurface remain straight and normal, both rotary inertia and shear deformation are neglected), strains and curvature change at the middle surface are

$$\varepsilon_0 = \frac{\partial u_0}{\partial x}, \ \kappa = -\frac{\partial^2 w}{\partial x^2} \tag{24}$$

where u, w are displacements in x and z directions, respectively. Normal strain at any point would be

$$\varepsilon = \varepsilon_0 + z\kappa \tag{25}$$

Force and moment resultants are calculated using

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix} \tag{26}$$

The equations of motion are

$$\frac{\partial^2 M}{\partial x^2} = -p_z \tag{27}$$

$$\frac{\partial N}{\partial x} = -p_x \tag{28}$$

The potential strain energy stored in a beam during elastic deformation is

$$PE = \frac{1}{2}\int_V \sigma\varepsilon dV = \frac{1}{2}\int_0^l \left(N\varepsilon_0 + M\kappa\right)dx \tag{29}$$

Writing this expression for every lamina and summing for the laminate we have

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\varepsilon_0\right)^2 + 2B_{11}\varepsilon_0\kappa + D_{11}\kappa^2\right)dx \tag{30}$$

Substituting kinematic relations (24) into equation (30), it will become

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\frac{\partial u_0}{\partial x}\right)^2 + 2B_{11}\left(\frac{\partial u_0}{\partial x}\right)\left(-\frac{\partial^2 w}{\partial x^2}\right) + D_{11}\left(-\frac{\partial^2 w}{\partial x^2}\right)^2\right)dx \tag{31}$$

The work done by external forces on the beam is

$$W = \frac{1}{2}\int_0^l \left(p_x u_0 + p_z w\right)dx \tag{32}$$

The kinetic energy for each lamina is

$$KE = \frac{1}{2}b\rho^{(k)}\int_0^l\int_{z_{k-1}}^{z_k}\left(\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right)dx \tag{33}$$

where $\rho^{(k)}$ is the lamina density per unit volume, and t is time. The kinetic energy of the entire beam is

$$KE = \frac{I_1}{2}\int_0^l\left(\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right)dx \tag{34}$$

where I_1 is the average mass density of the beam per unit length. These energy expressions can be used in an energy-based analysis like finite element or Ritz analyses.

Inserting constitutive equations in equations of motion (26) will result in

$$A_{11}\frac{\partial^2 u}{\partial x^2} - B_{11}\frac{\partial^3 w}{\partial x^3} + p(x) = 0 \tag{35}$$

$$B_{11}\frac{\partial^3 u}{\partial x^3} - D_{11}\frac{\partial^4 w}{\partial x^4} + q(x) = 0 \tag{36}$$

Solving these two equations for u and w will result in the following differential equations.

$$\left[\frac{A_{11}D_{11} - B_{11}^2}{A_{11}}\right]\frac{\partial^4 w}{\partial x^4} = q(x) - \frac{B_{11}}{A_{11}}\frac{\partial p(x)}{\partial x} \tag{37}$$

$$\left[\frac{A_{11}D_{11} - B_{11}^2}{B_{11}}\right]\frac{\partial^3 u_0}{\partial x^3} = q(x) - \frac{D_{11}}{B_{11}}\frac{\partial p(x)}{\partial x} \tag{38}$$

Stress in the axial direction in any lamina can be found by the following equation

$$\sigma_x = \overline{Q_{11}}\left(\varepsilon_0 + z\kappa\right) = \overline{Q_{11}}\left(\frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) \tag{39}$$

Different loading and boundary conditions can be applied to these equations in order to find equations for u and w. These boundary conditions are

Simply supported: $w = 0,\ M = 0$

Clamped: $w = 0,\ \dfrac{dw}{dx} = 0$

Free: $V = 0,\ M = 0$

where V and M are shear force and bending moment and are linearly dependent on third and second derivatives of w, respectively. Here, we propose solutions for both ends simply supported and both ends clamped with constant loading q_0. For specific case of

$q(x)=q_0$ with simply supported boundary conditions at both ends and assuming $u_0(0)=0$ we have

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\right]w(x)=\frac{q_0 l^4}{24}\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)\right] \tag{40}$$

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{B_{11}}\right]u_0(x)=\frac{q_0 l^3}{24}\left[4\left(\frac{x}{l}\right)^3-6\left(\frac{x}{l}\right)^2\right] \tag{41}$$

$$\sigma_x=\frac{q_0 l^2}{2\left(A_{11}D_{11}-B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2-\left(\frac{x}{l}\right)\right]\overline{Q_{11}}\left(B_{11}-zA_{11}\right) \tag{42}$$

$$\tau_{xz}=\frac{1}{b}\int_h^z\frac{\partial\sigma_x}{\partial x}dz=\frac{q_0 l}{2b\left(A_{11}D_{11}-B_{11}^2\right)}\left[2\left(\frac{x}{l}\right)-1\right]\int_h^z\overline{Q_{11}}\left(B_{11}-zA_{11}\right)dz \tag{43}$$

for clamped boundary conditions at both ends we have

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\right]w(x)=\frac{q_0 l^4}{24}\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)^2\right] \tag{44}$$

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{B_{11}}\right]u_0(x)=\frac{q_0 l^3}{24}\left[4\left(\frac{x}{l}\right)^3-6\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)\right] \tag{45}$$

$$\sigma_x=\frac{q_0 l^2}{2\left(A_{11}D_{11}-B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2-\left(\frac{x}{l}\right)+\frac{1}{6}\right]\overline{Q_{11}}\left(B_{11}-zA_{11}\right) \tag{46}$$

$$\tau_{xz}=\frac{1}{b}\int_h^z\frac{\partial\sigma_x}{\partial x}dz=\frac{q_0 l}{2b\left(A_{11}D_{11}-B_{11}^2\right)}\left[2\left(\frac{x}{l}\right)-1\right]\int_h^z\overline{Q_{11}}\left(B_{11}-zA_{11}\right)dz \tag{47}$$

One should note that for the simply supported boundary condition the maximum moment and consequently maximum stress occurs at the middle of the beam, while for the clamped case maximum stress occurs at the two ends.

Matrix approach (CBT)

Inserting the strain and curvature relations in the force and moment resultants equations and using those in the equations of motion, one can express the equations of motion in terms of displacements. Expressing those equations in matrix form we have

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} + \begin{bmatrix} p_x \\ -p_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{48}$$

where $L_{11} = A_{11} \frac{\partial^2}{\partial x^2}$, $L_{22} = D_{11} \frac{\partial^4}{\partial x^4}$, $L_{12} = L_{21} = -B_{11} \frac{\partial^3}{\partial x^3}$.

The beam is supposed to have simply supported boundary condition. So we have on x=0, l.

$$w_0 = N_x = M_x = 0 \tag{49}$$

The above equations of motion as well as boundary terms are satisfied if one chooses displacements as

$$[u, w] = \sum_{m=1}^{M} [A_m \cos(\alpha_m x), C_m \sin(\alpha_m x)] \tag{50}$$

where $\alpha_m = m\pi / l$ and l is the beam length. The external forces can be expanded in a Fourier series in x

$$[p_x, p_z] = \sum_{m=1}^{M} [p_{xm} \sin(\alpha_m x), p_{zm} \cos(\alpha_m x)] \tag{51}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix} = 0 \tag{52}$$

$$\begin{bmatrix} A_m \\ C_m \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} -p_{xm} \\ p_{zm} \end{bmatrix} \tag{53}$$

where $C_{11} = -\alpha_m^2 A_{11}$, $C_{22} = \alpha_m^4 D_{11}$, $C_{21} = -C_{12} = \alpha_m^3 B_{11}$. Stress in the axial direction would be found using the following procedure.

$$\begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}^{-1} \begin{bmatrix} N \\ M \end{bmatrix} \tag{54}$$

$$\sigma_x = \overline{Q_{11}} \left(\varepsilon_0 + z\kappa \right) \tag{55}$$

Shear Deformation Beam Theory

In this chapter a first order shear deformation theory (FSDT) approach is to account for shear deformation and rotary inertia.

$$u = u_0 + z\psi, \;\; w = w_0 \tag{56}$$

Strains and curvature changes at the middle surface are:

$$\varepsilon_0 = \frac{\partial u_0}{\partial x}, \; \kappa = \frac{\partial \psi}{\partial x}, \; \gamma = \frac{\partial w}{\partial x} + \psi \tag{57}$$

where ε_0 is middle surface strain, γ is the shear strain at the neutral axis and ψ is the rotation of a line element perpendicular to the original direction. Normal strain at any point can be found using equation 25. Force and moment resultants as well as shear forces are calculated using

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \kappa \\ \gamma \end{bmatrix} \tag{58}$$

where for A_{55} we have

$$A_{55}=\frac{5}{4}\sum_{k=1}^{N} b\overline{Q}_{55}^{k}\left[\left(h_k-h_{k-1}\right)-\frac{4}{3h^2}\left(h_k^3-h_{k-1}^3\right)\right] \tag{59}$$

The equations of motion considering rotary inertia and shear deformation are

$$\frac{\partial N}{\partial x}=-p_x \tag{60}$$

$$\frac{\partial Q}{\partial x}=-p_z \tag{61}$$

$$\frac{\partial M}{\partial x}-Q=0 \tag{62}$$

The potential strain energy stored in a beam during elastic deformation is

$$PE=\frac{1}{2}\int_V \sigma\varepsilon dV=\frac{1}{2}\int_0^l\left(N\varepsilon_0+M\frac{\partial\psi}{\partial x}+Q\gamma\right)dx \tag{63}$$

Writing this expression for every lamina and summing for the laminate we have

$$PE=\frac{1}{2}\int_0^l\left(A_{11}\left(\varepsilon_0\right)^2+2B_{11}\varepsilon_0\kappa+D_{11}\kappa^2+A_{55}\gamma^2\right)dx \tag{64}$$

Substituting kinematic relations into equation (64) it will become

$$PE=\frac{1}{2}\int_0^l\left(A_{11}\left(\frac{\partial u_0}{\partial x}\right)^2+2B_{11}\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right)+D_{11}\left(\frac{\partial\psi}{\partial x}\right)^2+A_{55}\left(\psi+\frac{\partial w}{\partial x}\right)^2\right)dx \tag{65}$$

The work done by external forces on the beam is found by equation (32). Finding the kinetic energy for each layer and then summing for all layers yield the kinetic energy of the entire beam.

$$KE=\int_0^l\left(I_1\left(\frac{\partial u_0}{\partial t}\right)^2+I_1\left(\frac{\partial w}{\partial t}\right)^2+2I_2\left(\frac{\partial u_0}{\partial t}\right)\left(\frac{\partial\psi}{\partial t}\right)+I_3\left(\frac{\partial\psi}{\partial t}\right)^2\right)dx \tag{66}$$

These energy expressions can be used in an energy-based analysis like finite element or Ritz analyses.

Euler Approach (SDBT)

Inserting constitutive equations in the equations of motion will result in

$$A_{11}\frac{\partial^2 u_0}{\partial x^2}+B_{11}\frac{\partial^2 \psi}{\partial x^2}+p_x(x)=0 \tag{67}$$

$$A_{55}\left(\frac{\partial \psi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+p_z\left(x\right)=0 \tag{68}$$

$$B_{11}\frac{\partial^2 u}{\partial x^2}+D_{11}\frac{\partial^2 \psi}{\partial x^2}-A_{55}\left(\psi+\frac{dw}{dx}\right)=0 \tag{69}$$

Taking, the second derivative of equation (68) and solving for $\frac{\partial^3 \psi}{\partial x^3}$ from equations (67, 69) will result in the following equations.

$$\frac{\partial^4 w}{\partial x^4}=\left[\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right]q(x)-\frac{1}{A_{55}}\left[\frac{\partial^2 q(x)}{\partial x^2}\right]-\left[\frac{B_{11}}{A_{11}D_{11}-B_{11}^2}\right]\frac{\partial p(x)}{\partial x} \tag{70}$$

$$\frac{\partial^3 u_0}{\partial x^3}=\left[\frac{B_{11}}{A_{11}D_{11}-B_{11}^2}\right]q(x)-\frac{1}{A_{11}}\frac{\partial p(x)}{\partial x} \tag{71}$$

$$\frac{\partial^3 \psi}{\partial x^3}=\left[\frac{B_{11}}{A_{11}D_{11}-B_{11}^2}\right]\frac{\partial p(x)}{\partial x}-\left[\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right]q(x) \tag{72}$$

For the specific case of $q(x)=q_0$ with simply supported boundary conditions we have

$$w(x)=\frac{q_0 l^4}{24}\left(\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right)\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)\right]+\frac{q_0 l^2}{2A_{55}}\left[\left(\frac{x}{l}\right)+\left(\frac{x}{l}\right)^2\right] \tag{73}$$

$$\psi(x)=\frac{q_0 l^3}{24}\left(\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right)\left[1-4\left(\frac{x}{l}\right)^3+6\left(\frac{x}{l}\right)^2\right]+\frac{q_0 l}{2A_{55}} \tag{74}$$

$$\sigma_x = \frac{q_0 l^2}{2\left(A_{11}D_{11} - B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)\right]\overline{Q_{11}}\left(B_{11} - zA_{11}\right) - \frac{2q_0 l^2 z}{A_{55}} \tag{75}$$

Maximum deflection occurs at the middle of the beam equal to

$$w_{\max} = \frac{5q_0 l^4}{384}\left(\frac{A_{11}}{A_{11}D_{11} - B_{11}^2}\right) + \frac{q_0 l^2}{8A_{55}} \tag{76}$$

The first term in equation (76) is deflection due to bending and the second term is due to shear. For the clamped boundary condition one can use the term due to bending from CBT analysis and add the term due to shear.

Matrix Approach (SDBT)

Equations of motion in terms of the displacements in matrix form are

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} + \begin{bmatrix} p_x \\ -p_z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{77}$$

where $L_{11} = A_{11}\frac{\partial^2}{\partial x^2}$, $L_{22} = -A_{55}\frac{\partial^2}{\partial x^2}$, $L_{33} = D_{11}\frac{\partial^2}{\partial x^2} - A_{55}$, $L_{13} = L_{31} = B_{11}\frac{\partial^2}{\partial x^2}$,

$L_{23} = L_{32} = -A_{55}\frac{\partial}{\partial x}$, $L_{12} = L_{21} = 0$. The following simply supported boundary conditions are used on x=0, l

$$w_0 = N_x = \frac{\partial \psi}{\partial x} = 0 \tag{78}$$

The above equations would be satisfied if

$$\left[u_0, w_0, \psi\right] = \sum_{m=1}^{m}\left[A_m \cos(\alpha_m x), C_m \sin(\alpha_m x), B_m \cos(\alpha_m x)\right] \tag{79}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} -p_{xm} \\ p_{zm} \\ 0 \end{bmatrix} \tag{80}$$

where $C_{11} = -\alpha_m^2 A_{11}$, $C_{22} = \alpha_m^2 A_{55}$, $C_{33} = -\alpha_m^2 D_{11} - A_{55}$, $C_{31} = C_{13} = -\alpha_m^2 B_{11}$, $C_{23} = -C_{32} = A_{55}\alpha_m$, $C_{21} = C_{12} = 0$.

Vibration Analysis

In this section, classic and FSDT beam models will be evaluated for their accuracy in a vibration analysis using different approaches for stiffness parameters calculation. Their results will be compared with those obtained using a 3D finite element model for different laminates (unidirectional, symmetric and asymmetric cross ply and symmetric and asymmetric angle-ply).

Classical Beam Theory

Equations of motion for dynamic analysis of laminated beams are

$$\frac{\partial^2 M}{\partial x^2} = I_1 \frac{\partial^2 w}{\partial t^2} - p_z \tag{81}$$

$$\frac{\partial N}{\partial x} = I_1 \frac{\partial^2 u}{\partial t^2} - p_x \tag{82}$$

where $I_1 = \sum_{k=1}^{N} b\rho^{(k)}\left(h_k - h_{k-1}\right)$. Expressing those equations in matrix form we have for free vibration

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} + \begin{bmatrix} -I_1 & 0 \\ 0 & I_1 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{83}$$

The equations of motion as well as simply supported boundary terms are satisfied if one chooses displacements as

$$[u, w] = \sum_{m=1}^{M} [A_m \cos(\alpha_m x), C_m \sin(\alpha_m x)] \sin(\omega t) \tag{84}$$

Substituting equation 84 in the equations of motion the characteristic equation is

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \omega^2 \begin{bmatrix} I_1 & 0 \\ 0 & -I_1 \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix} = 0 \tag{85}$$

The nontrivial solution for natural frequency can be found by setting the determinant of the characteristic equation of matrix to zero.

One should note here that if the laminate is symmetric, the B_{11} term vanishes and the bending frequencies are totally decoupled from axial ones. As a result, the following well-known formula for the natural frequencies of a symmetrically laminated simply supported composite beam is

$$\omega_n = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{D_{11}}{\rho A}} \tag{86}$$

where ρ is the density, ℓ is length and A is the cross sectional area of the beam. As we will see later it cannot be used for thick laminates and those that have any kind of coupling.

Shear Deformation Beam Theory

The equations of motion considering rotary inertia and shear deformation are

$$\frac{\partial N}{\partial x} = I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^2 \psi}{\partial t^2} - p_x \tag{87}$$

$$-\frac{\partial Q}{\partial x} = p_z - I_1 \frac{\partial^2 w}{\partial t^2} \tag{88}$$

$$\frac{\partial M}{\partial x} - Q = I_2 \frac{\partial^2 u}{\partial t^2} + I_3 \frac{\partial^2 \psi}{\partial t^2} \tag{89}$$

where $\left(I_1, I_2, I_3\right) = \sum_{k=1}^{N} b\rho^{(k)} \left(\left(h_k - h_{k-1}\right), \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right), \frac{1}{3}\left(h_k^3 - h_{k-1}^3\right) \right)$. So by expressing the equations of motion in terms of displacement we have in matrix form (for free vibration)

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} - \begin{bmatrix} I_1 & 0 & I_2 \\ 0 & -I_1 & 0 \\ I_2 & 0 & I_3 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{90}$$

Further mathematical manipulation enables one to express the equations of motion in terms of displacements. The following boundary conditions can be applied to the beam.

Simply supported (S1): $w = N = M = 0$ (91)
Simply supported (S2): $w = u = M = 0$ (92)
Clamped (C1): $N = w = \psi = 0$ (93)
Clamped (C2): $u = w = \psi = 0$ (94)
Free (F1): $N = M = Q = 0$ (95)
Free (F2): $u = M = Q = 0$ (96)

Exact solution is available for the simply supported S1 boundary conditions where we can assume

$$\left[u_0, w_0, \psi\right] = \sum_{m=1}^{m} \left[A_m \cos(\alpha_m x), C_m \sin(\alpha_m x), B_m \cos(\alpha_m x)\right] \sin(\omega t) \tag{97}$$

For other boundary conditions, the analysis is not this straightforward and a numerical solution is usually employed. In this chapter we deal with the exact solution for S1-S1 as well as numerical solutions for C2-C2 and C2-F1 (completely clamped-completely free) boundary conditions. Using equation 97 in equation 90 will give

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \omega^2 \begin{bmatrix} I_1 & 0 & I_2 \\ 0 & -I_1 & 0 \\ I_2 & 0 & I_3 \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{nm} \\ 0 \end{bmatrix} = 0 \tag{98}$$

The nontrivial solution for natural frequency can be found by setting the determinant of characteristic equation matrix to zero.

For boundary conditions other than simply supported, the General Differential Quadrature (GDQ) method is used to solve the equations. The method has been used by several investigators [22, 23]. In this method, the derivative of a function is approximated by a weighted linear sum of the function values at all the discrete points.

$$\frac{dF^{(n)}(x_i)}{dx} = \sum_{j=1}^{N} C_{ij}^{(n)} F\left(x_j\right) \tag{99}$$

Generally, by considering the Lagrange interpolation polynomials as test functions, the coefficients for the first order derivative become

$$C_{ij,i\neq j}^{(1)} = \frac{M(x_i)}{(x_i - x_j) M(x_j)} \tag{100}$$

$$M(x_i) = \prod_{j=1, j\neq i}^{N} (x_i - x_j) \tag{101}$$

for higher order derivatives

$$C_{ij,i\neq j}^{(n)} = n\left(C_{ij}^{(1)}.C_{ii}^{(n-1)} - C_{ij}^{(n-1)} / \left(x_i - x_j\right)\right) \tag{102}$$

and for all derivatives

$$C_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} C_{ij}^{(n)} \tag{103}$$

The sampling points are selected based on the Chebyshev–Gauss–Lobatto (C–G–L) grid distribution to get denser population near boundaries.

$$x_i = \frac{a}{2}\left[1-\cos\left(\frac{\pi(i-1)}{N-1}\right)\right] \tag{104}$$

Application of this method to equations of motion with respect to displacement functions will give a system of equations in matrix form that in this chapter are solved using MATLAB. Then, the results for static and free vibration are compared with those obtained using a 3D finite element model for different laminates (unidirectional, symmetric and asymmetric cross ply and symmetric and asymmetric angle-ply).

Case study

A rectangular cross section beam model having 1 m length, 0.025 m width, and 0.05 m height was made in ANSYS finite element code. Solid elements were used to apply 3D elasticity. A convergence study was done and the convergent model had 8 elements in thickness, 4 elements in width direction and 160 elements in length direction comprising of more than 5000 elements. Ratio of length to height of 20 was selected to be at the border of thin beams. Figure 3.3 shows the model in ANSYS® software.

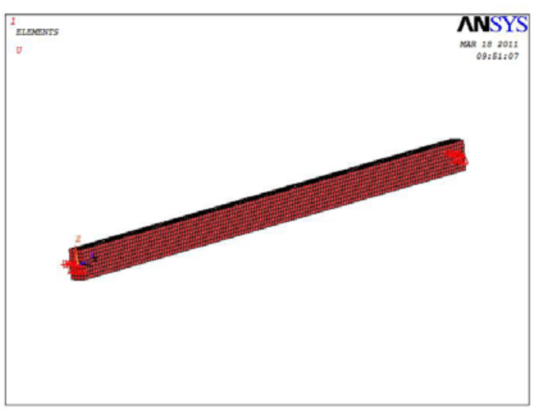

Figure 3.3 FEM model in ANSYS software

The simply supported boundary condition was modeled by applying constraints on the z direction at midline of the end faces. The material properties are

Graphite/Epoxy, E_1 = 138 GPa, E_2 =8.96 GPa, υ_{12}=0.3, G_{12}= 7.1 GPa, ρ =1580 kg/m^3, a/h = 20, b/h =0.5, E_1/E_2 = 15.4, G_{12}/E_2 = 0.79, υ_{12} = 0.3

Results and Discussion

Different stiffness parameters for calculating of deflection and natural frequencies were evaluated. Five different laminates were selected to cover different kinds of composite beams with rectangular cross section. These include unidirectional, symmetric cross-ply, asymmetric cross-ply, angle-ply and general laminates. The results for nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_z l^4\right)$ and natural frequencies $\Omega = \omega l^2 \sqrt{12\rho / E_1 h^2}$ of the first 5 modes for different boundary conditions are given in Tables 3.2 through 3.8. The deflection using $(S_{11})_f$ is not presented because of ill conditioning. The average error in all cases is around 1 %.

Table 3.2 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_z l^4\right)$ of rectangular graphite/epoxy simply supported beam using matrix approach (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

Laminate	CBT			SDBT			3D FEM
	(S_{11})	**$(S_{11})_e$**	**$(S_{11})_c$**	**(S_{11})**	**$(S_{11})_e$**	**$(S_{11})_c$**	
$[0]_4$	1.566	1.575	1.575	1.728	1.739	1.739	1.723
$[0/90]_s$	1.773	1.784	1.778	1.937	1.948	1.941	1.923
$[0_2/90_2]$	6.975	7.016	6.997	7.138	7.179	7.160	6.965
$[45]_4$	4.789	13.88	13.88	4.951	14.04	14.04	12.79
$[30_2/60_2]$	5.798	12.96	11.80	5.961	13.13	11.96	12.50

Table 3.3 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_z l^4\right)$ of rectangular graphite/epoxy simply supported beam using Euler approach (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

Laminate	CBT			SDBT			3D FEM
	(S_{11})	**(S_{11})$_e$**	**(S_{11})$_c$**	**(S_{11})**	**(S_{11})$_e$**	**(S_{11})$_c$**	
[0]$_4$	1.554	1.563	1.563	1.704	1.713	1.713	1.723
[0/90]$_s$	1.758	1.770	1.763	1.910	1.920	1.913	1.923
[0_2/90_2]	6.917	6.417	6.940	7.067	6.567	7.090	6.965
[45]$_4$	4.749	13.77	13.77	4.899	13.92	13.92	12.79
[30_2/60_2]	5.749	12.74	11.70	5.900	12.89	11.85	12.50

Table 3.4 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_z l^4\right)$ of rectangular graphite/epoxy clamped-clamped beam using Euler approach (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

Laminate	**CBT**			**SDBT**			**FEM**
	(S_{11})	**(S_{11})$_e$**	**(S_{11})$_c$**	**(S_{11})**	**(S_{11})$_e$**	**(S_{11})$_c$**	**3D**
[0]$_4$	0.3107	0.3126	0.3126	0.4611	0.4630	0.4630	0.4585
[0/90]$_s$	0.3517	0.3538	0.3526	0.5021	0.5042	0.5030	0.4934
[45]$_4$	0.9498	2.753	2.753	1.100	2.903	2.903	2.853
[0/90]$_s$	1.383	1.283	6.940	1.534	1.434	7.090	1.542
[30_2/60_2]	1.151	2.548	11.70	1.301	2.698	11.85	2.903

Table 3.5 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_z l^4\right)$ of rectangular graphite/epoxy clamped-clamped and clamped-free beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

Laminate		**[0]$_4$**	**[0/90]$_s$**	**[0_2/90_2]**	**[45]$_4$**	**[30_2/60_2]**
CC	FSDT (ABD)$_e$	0.4629	0.5043	1.542	2.721	2.903
	3D FEM	0.4604	0.4966	1.563	2.649	2.855
	Error (%)	0.54	1.54	-1.36	2.74	1.71
CF	FSDT (ABD)$_e$	15.60	17.59	67.40	124.0	132.7
	3D FEM	15.59	17.53	67.39	121.9	131.8
	E (%)	0.05	0.30	0.00	1.74	0.69

Table 3.6 Nondimensional natural frequencies $\left(\Omega = \omega l^2 \sqrt{12\rho / E_1 h^2}\right)$ of rectangular graphite/epoxy simply supported beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3)

				$[0]_4$					
n	CBT				SDBT				3D FEM
	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	
1	9.898	9.869	9.869	9.869	9.431	9.406	9.406	9.406	9.373
2	39.59	39.48	39.48	39.48	33.41	33.34	33.34	33.34	32.98
3	89.08	88.82	88.82	88.82	64.53	64.43	64.43	64.43	63.28
4	158.4	157.9	157.9	157.9	98.11	98.00	98.00	98.00	95.77
5	247.5	246.7	246.7	246.7	132.2	132.1	132.1	132.1	128.7
				$[0/90]_s$					
n	CBT				SDBT				3D FEM
	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	
1	9.302	9.275	9.291	9.291	8.910	8.886	8.901	8.901	8.873
2	37.21	37.10	37.16	37.16	31.93	31.86	31.90	31.90	31.65
3	83.72	83.47	83.62	83.62	62.37	62.27	62.33	62.33	61.51
4	148.8	148.4	148.7	148.7	95.66	95.54	95.61	95.61	93.91
5	232.5	231.9	232.3	232.3	129.7	129.6	129.7	129.7	126.9
				$[0_2/90_2]$					
n	CBT				SDBT				FEM
	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	(ABD)	$(ABD)_e$	$(ABD)_k$	$(ABD)_c$	3D
1	4.688	4.674	NA	4.680	4.637	4.810	NA	4.630	4.61
2	18.72	18.66	NA	18.69	17.95	18.58	NA	17.92	17.65
3	41.99	41.87	NA	41.92	38.41	39.62	NA	38.36	37.25
4	74.34	74.12	NA	74.22	64.16	65.94	NA	64.09	61.35
5	115.5	115.2	NA	115.3	93.49	95.73	NA	93.40	88.30

Table 3.6 (continued)

[45]$_4$									
n	CBT				SDBT				FEM
	(ABD)	(ABD)$_e$	(ABD)$_k$	(ABD)$_c$	(ABD)	(ABD)$_e$	(ABD)$_k$	(ABD)$_c$	3D
1	5.661	3.325	4.540	4.540	5.566	3.303	4.489	4.489	3.354
2	22.65	18.16	13.30	18.16	21.23	12.96	17.38	17.38	12.97
3	50.95	40.86	29.93	40.86	44.56	28.30	37.24	37.24	28.32
4	90.58	72.64	53.20	72.64	72.92	48.41	62.29	62.29	48.32
5	141.5	113.5	83.13	113.5	104.3	72.32	90.94	90.94	71.93
[30$_2$/60$_2$]									
n	CBT				SDBT				FEM
	(ABD)	(ABD)$_e$	(ABD)$_k$	(ABD)$_c$	(ABD)	(ABD)$_e$	(ABD)$_k$	(ABD)$_c$	3D
1	5.143	3.440	NA	4.801	5.073	3.433	NA	4.745	3.483
2	20.55	13.76	NA	19.18	19.51	13.46	NA	18.33	13.46
3	46.18	30.93	NA	43.06	41.37	29.34	NA	39.13	29.22
4	81.92	54.93	NA	76.31	68.43	50.10	NA	65.19	49.86
5	127.6	85.7	NA	118.7	98.8	74.69	NA	94.76	73.97

Table 3.7 Nondimensional natural frequencies $\left(\Omega = \omega l^2 \sqrt{12\rho / E_1 h^2}\right)$ of rectangular graphite/epoxy clamped-clamped beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ=1580 kg/m^3)

CC	[0]$_4$		[0/90]$_s$		[0$_2$/90$_2$]		[45]$_4$		[30$_2$/60$_2$]	
Mode	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM
1	18.26	18.32	17.50	17.65	10.04	9.972	7.325	7.389	7.565	7.669
2	42.50	42.78	41.21	41.78	26.04	25.70	19.52	19.68	20.11	20.36
3	71.53	72.25	69.84	71.10	73.08	71.39	36.72	36.98	37.73	38.22
4	102.8	104.2	100.9	103.0	101.2	98.54	57.89	58.26	59.32	59.94
5	135.3	137.4	133.2	136.4	131.2	127.4	82.17	82.61	84.00	84.74
E(%)	0.96		1.68		-2.01		0.71		1.16	

Table 3.8 Nondimensional natural frequencies $\left(\Omega = \omega l^2 \sqrt{12\rho / E_1 h^2}\right)$ of rectangular graphite/epoxy clamped-free beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3)

CF	**[0]$_4$**		**[0/90]$_s$**		**[0$_2$/90$_2$]**		**[45]$_4$**		**[30$_2$/60$_2$]**	
Mode	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM	(ABD)$_e$	FEM
1	3.435	3.437	3.237	3.242	1.656	1.657	1.181	1.189	1.221	1.237
2	19.08	19.12	18.20	18.31	10.03	9.986	7.267	7.314	7.507	7.592
3	46.31	46.51	44.62	45.10	26.72	26.44	19.79	19.91	20.40	20.61
4	77.84	78.39	75.67	76.78	49.10	48.28	37.35	37.51	38.37	38.73
5	111.3	112.4	108.9	110.8	75.41	73.79	59.06	59.29	60.58	61.06
E(%)	0.47		1.02		-1.07		0.56		1.02	

The deflection results show that using CBT is not even accurate for slenderness ratio of 20. Using FSDT along with normal definition for ABD terms is only accurate for cross-ply laminates. However, using equivalent ABDs corrects the results for other laminates having different couplings.

The natural frequencies results show that the classic beam model using normal ABD parameters is only valid for 1st mode of cross-ply laminates. The effective length becomes less on higher modes and the thin beam assumption no longer applies leading to inaccurate results. Although the [45]$_4$ laminate is symmetric; it has bending twisting coupling and using the normal ABD formulation leads to inaccurate results.

The (ABD)$_e$ equivalent stiffness parameters improve the classic approach for unsymmtric laminates but still not accurate for higher modes since the shear deformation is not included.

Using FSDT approach for thick beams along with $(ABD)_e$ (Eqs. 48-50) one can reach accurate results for higher modes. This approach does not have coupling problems and accurate results for all laminates are achieved. The overall range of error is about 1 percent. The other defined equivalent parameters by compliance matrix are not as accurate as $(ABD)_e$ and even do not have real results in some cases.

The mode shapes for the first five modes of simply supported beam with laminate $[30_2/60_2]$ is presented in the following figures. One should note the twist induced in the mode shapes due to bending-twisting coupling.

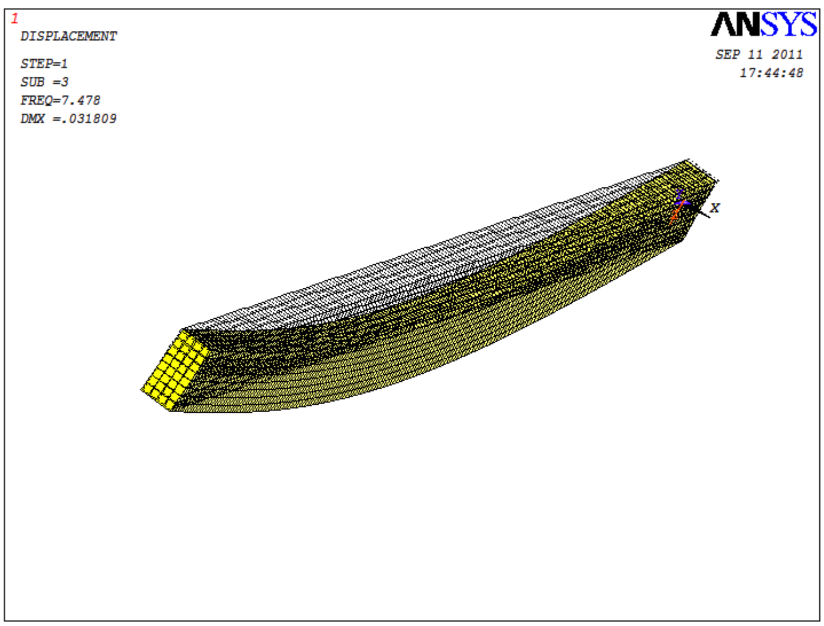

Figure 3.4 Mode shape number 1 of the straight beam with $[30_2/60_2]$ laminate

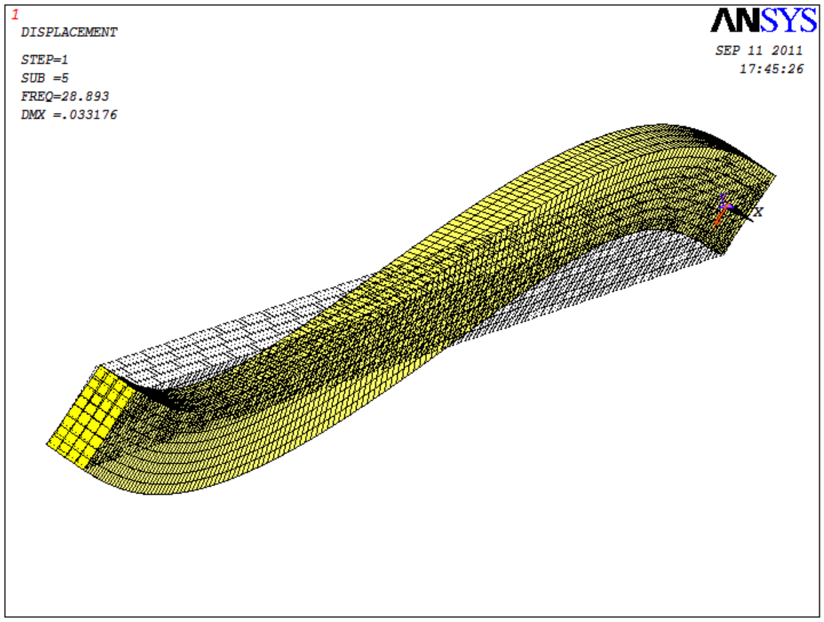

Figure 3.5 Mode shape number 2 of the straight beam with $[30_2/60_2]$ laminate

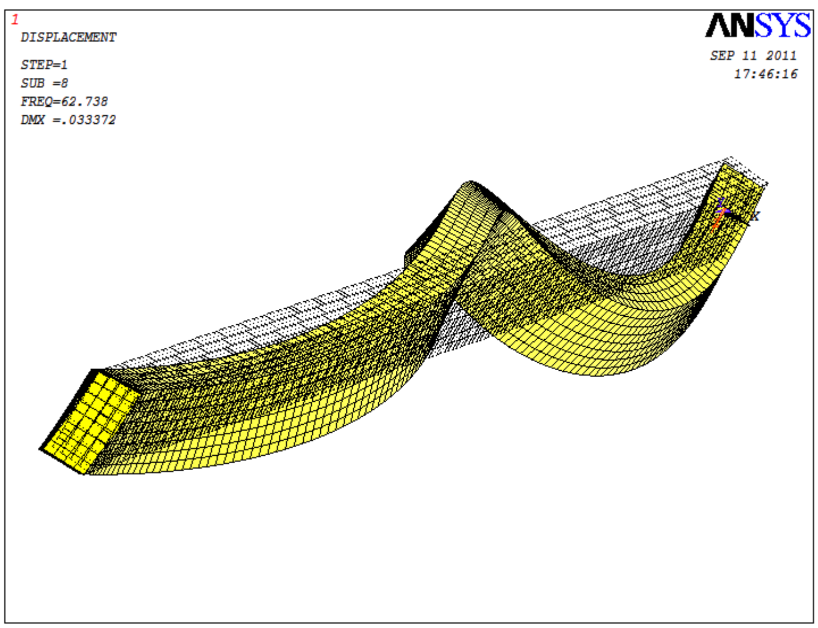

Figure 3.6 Mode shape number 3 of the straight beam with $[30_2/60_2]$ laminate

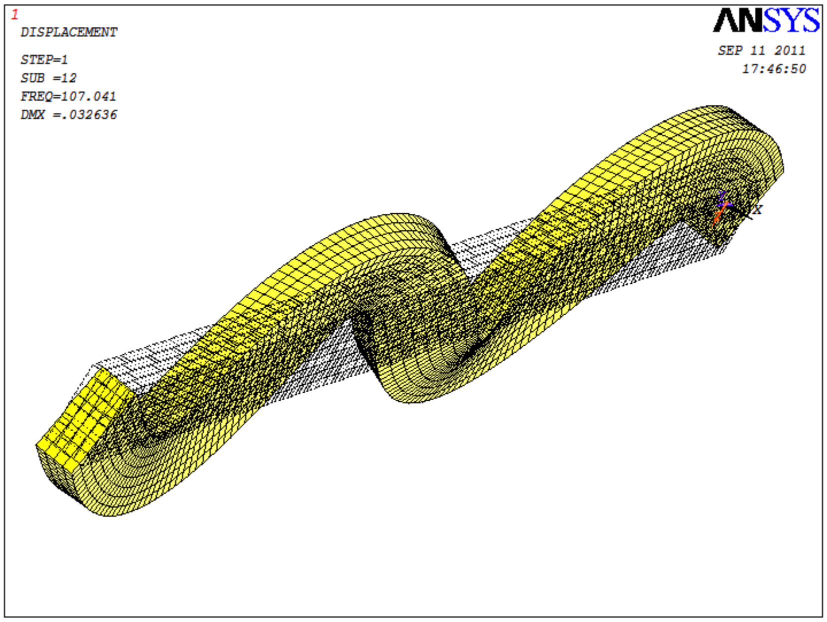

Figure 3.7 Mode shape number 4 of the straight beam with [30_2/60_2] laminate

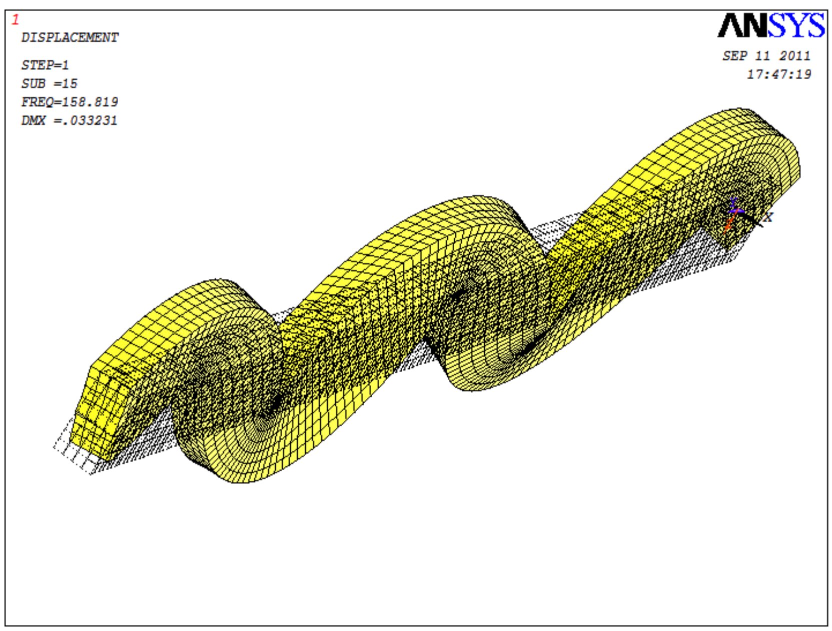

Figure 3.8 Mode shape number 5 of the straight beam with [30_2/60_2] laminate

In another case, the results provided by Jun et al. [26] that considers bending-twisting coupling has been compared with the present method. The first case is a glass-polyester simply supported unidirectional beam with [45] laminate. The material properties are as follows:

$E_1=37.41*10^9$, $E_2=13.67*10^9$, $G_{12}=5.478*10^9$, $G_{13}=6.03*10^9$, $G_{23}=6.666*10^9$ (Pa)

$\upsilon_{12}=0.3, \rho=1968.9\left(kg/m^3\right)$, l=0.11179, b=12.7*10^{-3}, h=3.38*10^{-3} (m)

The second case is a graphite-epoxy simply supported beam with [30/50/30/50] layup. The material properties are as follows:

$E_1=144.8*10^9$, $E_2=9.65*10^9$, $G_{12}= G_{13}=4.14*10^9$, $G_{23}=3.45*10^9$ (Pa)

$\upsilon_{12}=0.3, \rho=1389.23\left(kg/m^3\right)$, l=0.381, b=25.4*10^{-3}, h=25.4*10^{-3} (m)

The results of first five natural frequencies are presented in Table 3.9.

Table 3.9 Comparison of the natural frequencies (Hz) in present method to Ref [26]

Mode	Case 1		Case 2		
	Ref [26]	present	Ref [26]	present	3D FEM
1	348.1	338.2	353.7	252.4	275.6
2	1346.2	1347.3	1114.3	992.6	1018
3	3017.6	3010.9	2434.2	2171.9	2181
4	5285.3	5303.1	3450.3	3717	3627
5			4264.5	5542.8	5257

The results show that since bending-twisting coupling is considered in Ref [26] the results in case 1 match well with the present method. However, case 2 is a general laminate with all couplings present and comparison of the result in Ref [26] with 3D

FEM shows that it has considerable error. But the present method accurately predicts the natural frequencies.

Conclusion

A modified FSDT model was proposed for static and dynamic analysis of composite beams. The model accounts for various laminate couplings and shear deformation and rotary inertia. It is compared to different approaches and 3D FEM model. The results showed good accuracy of the model for rectangular beams. This model provides an accurate approach for calculating the natural frequencies of beams with arbitrary laminate for engineers and scientists.

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CHAPTER IV

STATIC AND VIBRATION ANALYSES OF THICK GENERALLY LAMINATED DEEP CURVED BEAMS WITH DIFFERENT BOUNDARY CONDITIONS

Introduction

Composite materials offer higher strength and stiffness to weight ratios than most metallic materials and have the possibility to tailor the design for specific purposes using different laminations. Among composite structures, curved beam components constitute a frequently encountered structural element of aerospace, marine and other structures.

Since all beam theories are based on three dimensional (3D) elasticity, 3D-based analyses (including finite element analyses, FEA) are the most accurate analytical ones. However, 3D FEA is a very expensive procedure demanding expensive machines and/or longer computational times if used for large scale structures. This is particularly the case when one dimension is significantly larger than the other two. For these structures, beam models are very efficient provided that they are built on accurate models and are verified against accurate 3D analyses. Several researchers have worked on the analysis of composite beam structures. Kapania and Raciti [1,2] published a review on advances in the analysis of laminated beams and plates. Rosen [3] reviewed the research on static, dynamic, and stability analysis of pretwisted rods and beams. Chidamparam and Leissa [3] reviewed the published literature on the vibrations of curved bars, beams, and rings of arbitrary shape which lie in a plane. Qatu [5] dedicated a book to vibration of composite beams, plates and shells. Hodges [6] made an extensive literature survey of the modern history of beam analysis. Recently, a review is conducted on laminated beams by the authors [7]. Among the conclusions of this review is the lack of a consistent set of equations that treat moderately thick beams with curvature and general lamination sequence.

This chapter is concerned with the development of a general approach for static and in-plane free vibration analyses of generally laminated composite deep curved beams. While the vibration part was treated before, the formulation there was limited to simply supported beams with cross-ply laminates [8]. In this chapter accurate deflection, moment resultants and natural frequencies of generally laminated deep curved beams are presented and can be used for possible use by researchers and practicing engineers.

Complexities in analysis of laminated curved beams

There are three complexities in the analysis of composite curved thick beams: shear deformation, deepness, and material couplings. These effects are not considered simultaneously in the literature. Moreover, for coupling problem, there has not been a method that considers all kind of couplings. In this chapter a simple method is proposed that considers all these problems.

Effect of shear deformation

The inclusion of shear deformation in the analysis of beams was first made by Timoshenko [9] and this inclusion has been incorporated by most researchers dealing with composite materials. This effect is higher in composite materials because structural or machine components made of composite materials are generally thicker and more flexible in shear than metallic materials. Consequently, the assumption of classical Euler-Bernoulli theory is usually inaccurate. In this chapter a first order shear deformation theory (FSDT), will be used.

Effect of deepness in constitutive equations

The geometry of the beam is depicted in Figure 4.1. The deepness or curvature complexity comes from the term (1+z/R) in the kinematic relations for strain. This term is either neglected (shallow beam theories) [10, 11], expanded [12, 13] or exactly integrated [5, 8] into the equations. Qatu [5, 8] showed that this term should be included in the analysis especially when the ratio of the span length to radius in a static analysis or the half sine wave length to radius ratio in a vibration analysis is more than 0.5. For higher ratios, shallow beam theory becomes inaccurate and a deep beam theory should be used. In this chapter we use the exact integration of the term into the ABD parameters.

$$A_{11} = R\sum_{k=1}^{N} b\overline{Q}_{11}^{(k)} \ln\left(\frac{R+z_k}{R+z_{k-1}}\right) \tag{1}$$

$$B_{11} = R\sum_{k=1}^{N} b\overline{Q}_{11}^{(k)} \left[\left(z_k - z_{k-1}\right) - R\ln\left(\frac{R+z_k}{R+z_{k-1}}\right)\right] \tag{2}$$

$$D_{11} = R\sum_{k=1}^{N} bQ_{11}^{(k)} \left[\frac{1}{2}\left(\left(R+z_k\right)^2 - \left(R+z_{k-1}\right)^2\right) - 2R\left(z_k - z_{k-1}\right) - R^2\ln\left(\frac{R+z_k}{R+z_{k-1}}\right)\right] \tag{3}$$

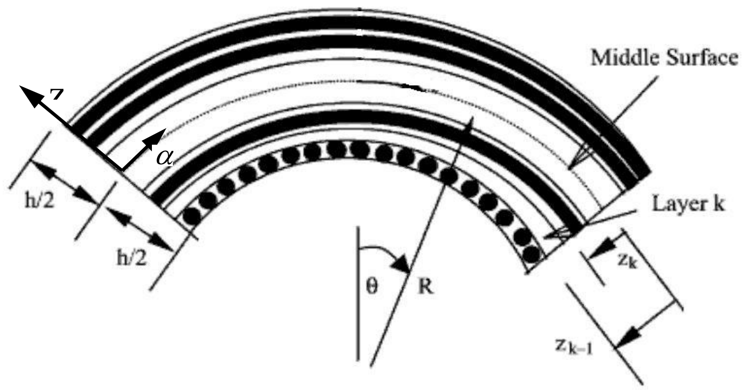

Figure 4.1 Geometry of the curved Beam.

Material coupling

The coupling complexity arises from the lamination sequence. In some applications such as aerospace structures, some parts are very weight limited and/or a specific deformation under loading may be desired (aeroelastic tailoring). In such applications the laminate usually has one of the material couplings that should be taken into account in the analysis of beam structures. These couplings include bending-twisting couplings, extension-bending coupling and extension-shearing couplings.

The only type of laminate that does not have any of the aforementioned couplings is symmetric cross-ply. For other laminates, a kind of coupling exists and introduces complexities in reducing the 3D problem into a one dimensional (1D). In 1D analysis of beams, 18 parameters in the ABD stiffness matrix are reduced to only 3 parameters (A_{11}, B_{11}, D_{11}) and only one term of the bending-stretching coupling is taken into account (i.e. B_{11}). In beams with layup other than symmetric cross ply, other coupling terms are not zero and ignoring them will result in inaccurate analyses. This problem raises the need to redefine the stiffness parameters so that the couplings can be considered in these three stiffness terms. This complexity has been shown to be very important and introduces errors in the analysis of beams [14]. Most of the papers published so far are limited to special laminates. Among FSDT works, some were validated for isotropic [15], or symmetric cross-ply laminates that have no coupling [16-20], or cross-ply laminates that have only bending-stretching coupling [8, 21]. In some other models, unidirectional beams with only bending-twisting coupling [22-26] were considered. A formulation for bending stiffness was proposed by Boay and Wee [14] and was applied to symmetric and asymmetric angle ply (bending-twisting coupling) using Euler-Bernoulli beam theory.

Higher order shear deformation theories (HSDT) were developed for composite beams to address issues of cross sectional warping and transverse normal strains. Most of them were validated for cross-ply laminates [11, 27-29]. Kapuria et al. [30] used zigzag theory to satisfy continuity of transverse shear stress through the laminate and showed to be accurate for natural frequency calculations of beams with specific geometry and lay-up (symmetric or cross-ply laminates). Another theory was global-local higher order theory proposed for beams by Zhen and Wanji [31] that was validated for cross-ply laminates.

Another problem that appears in asymmetric laminates is that in-plane and out-of-plane vibrations are coupled and those equations must be solved together. But as Bhimaraddi [15] showed, this has a very little effect for in-plane flexural vibrations. However, for out-of-plane or more importantly extensional modes, this effect is high and the equations should be solved coupled together.

Hajianmaleki and Qatu [32, 33] showed that by using equivalent modulus of elasticity of each lamina, one can get accurate results for static and dynamic analyses of generally laminated straight beams with any kind of coupling. The equivalent modulus of elasticity of each lamina is found using equation 4.

$$\frac{1}{E_x^k} = \frac{\cos^4(\theta_k)}{E_{11}} + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_{11}}\right)\cos^2(\theta_k)\sin^2(\theta_k) + \frac{\sin^4(\theta_k)}{E_{22}} \tag{4}$$

Using modified stiffness parameter for curved beam, we have for ABD terms

$$A_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \ln\left(\frac{R+z_k}{R+z_{k-1}}\right) \tag{5}$$

$$B_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \left[(z_k - z_{k-1}) - R\ln\left(\frac{R+z_k}{R+z_{k-1}}\right)\right] \tag{6}$$

$$D_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \left[\frac{1}{2}\left((R+z_k)^2 - (R+z_{k-1})^2\right) - 2R(z_k - z_{k-1}) - R^2\ln\left(\frac{R+z_k}{R+z_{k-1}}\right)\right] \tag{7}$$

In order to consider all complexities in laminated deep beams analysis, the equations 5-7 are proposed in this chapter to be used in analysis of laminated curved beams.

Analysis of laminated curved beams

To accommodate for shear deformation, a FSDT is proposed for generally laminated curved beams. The basic equations of such theory are described next.

Kinematic relations

For a curved beam, we have the following equations for displacements, curvature changes and strains.

$$u = u_0 + z\psi, \quad w = w_0 \tag{8}$$

$$\varepsilon_0 = \frac{\partial u_0}{\partial \alpha} + \frac{w_0}{R}, \quad \kappa = \frac{\partial \psi}{\partial \alpha}, \quad \gamma = \frac{\partial w}{\partial \alpha} + \psi - \frac{u}{R} \tag{9}$$

where u, and w are the displacements in the α and z directions; respectively (see Fig. 4.1). ε_0 is the middle surface strain, γ is the shear strain at the neutral axis and ψ is the rotation of a line element perpendicular to the original direction; respectively. Normal strain at any point is

$$\varepsilon = \frac{1}{\left(1 + \frac{z}{R}\right)}(\varepsilon_0 + z\kappa) \tag{10}$$

Force and moment resultants as well as shear forces are calculated using

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \kappa \\ \gamma \end{bmatrix} \tag{11}$$

where for A_{55} we have [5]

$$A_{55} = \frac{5}{4} \sum_{k=1}^{N} b \overline{Q}_{55}^{k} \left[\left(h_k - h_{k-1}\right) - \frac{4}{3h^2} \left(h_k^3 - h_{k-1}^3\right) \right] \tag{12}$$

Equations of motion

The equations of motion are [5]

$$\frac{\partial N}{\partial \alpha} + \frac{Q}{R} = \overline{I}_1 \frac{\partial^2 u}{\partial t^2} + \overline{I}_2 \frac{\partial^2 \psi}{\partial t^2} - p_\alpha \tag{13}$$

$$\frac{N}{R} - \frac{\partial Q}{\partial \alpha} = p_n - \overline{I}_1 \frac{\partial^2 w}{\partial t^2} \tag{14}$$

$$\frac{\partial M}{\partial \alpha} - Q = \overline{I}_2 \frac{\partial^2 u}{\partial t^2} + \overline{I}_3 \frac{\partial^2 \psi}{\partial t^2} \tag{15}$$

where

$$\left(\overline{I_1}, \overline{I_2}, \overline{I_3}\right) = \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}, I_2 + \frac{I_3}{R}, I_3 \right) \tag{16}$$

$$\left(I_1, I_2, I_3\right) = \sum_{k=1}^{N} b\rho^{(k)}\left(\left(h_k - h_{k-1}\right), \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right), \frac{1}{3}\left(h_k^3 - h_{k-1}^3\right) \right) \tag{17}$$

For each of the possible special cases, certain parameters will not be needed. For example, in a static analysis, the dynamic terms including derivation to time are removed. In case of a straight beam, the terms including R are removed. The loading in a normal free vibration analysis is ignored. Expressing equations of motion in terms of displacement we have in matrix form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} - \begin{bmatrix} \overline{I_1} & 0 & \overline{I_2} \\ 0 & -\overline{I_1} & 0 \\ \overline{I_2} & 0 & \overline{I_3} \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} = \begin{bmatrix} -p_\alpha \\ p_n \\ 0 \end{bmatrix} \tag{18}$$

where $L_{11} = A_{11}\dfrac{\partial^2}{\partial \alpha^2} - \dfrac{A_{55}}{R^2}$, $L_{22} = \dfrac{A_{11}}{R^2} - A_{55}\dfrac{\partial^2}{\partial \alpha^2}$, $L_{33} = D_{11}\dfrac{\partial^2}{\partial \alpha^2} - A_{55}$, $L_{12} = \left(\dfrac{A_{11} + A_{55}}{R}\right)\dfrac{\partial}{\partial \alpha}$

$L_{13} = B_{11}\dfrac{\partial^2}{\partial \alpha^2} + \dfrac{A_{55}}{R}$, $L_{23} = \left(\dfrac{B_{11}}{R} - A_{55}\right)\dfrac{\partial}{\partial \alpha}$, $L_{31} = L_{13}$, $L_{32} = L_{23}$, $L_{21} = L_{12}$

Boundary conditions

The following boundary conditions can be applied to the beam. These conditions would be applied on α =0, a.

Simply supported (S1): $w = N = M = 0$ (19a)

Simply supported (S2): $w = u = M = 0$ (19b)

Clamped (C1): $N = w = \psi = 0$ (20a)

Clamped (C2): $u = w = \psi = 0$ (20b)

Free (F1): $N = M = Q = 0$ (21a)

Free (F2): $u = M = Q = 0$ (21b)

Exact solution is available for the simply supported S1 boundary conditions [35] where we can assume

$$[u_0, w_0, \psi] = \sum_{m=1}^{m}[A_m \cos(\alpha_m \alpha), C_m \sin(\alpha_m \alpha), B_m \cos(\alpha_m \alpha)]\sin(\omega t) \tag{22}$$

where $\alpha_m = m\pi / a$ and a is the beam length. The term $\sin(\omega t)$ in equation 22 is only employed in vibration analysis. For other boundary conditions, the analysis is not this straightforward and a numerical solution is usually employed. In this chapter we deal with exact solution for S1-S1 as well as numerical solutions for C2-C2 and C2-F1 (cantilevered; completely clamped-completely free) boundary conditions.

For the simply supported boundary condition, the loading is expanded using Fourier transform

$$[p_\alpha, p_n] = \sum_{m=1}^{M}\left[p_{\alpha m} \cos(\alpha_m \alpha), p_{nm} \sin(\alpha_m \alpha)\right] \tag{23}$$

Substituting these equations in the equations of motion, the solution for static analysis

$$\begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} -p_{\alpha m} \\ p_{nm} \\ 0 \end{bmatrix} \tag{24}$$

and for free vibration analysis

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \omega^2 \begin{bmatrix} \overline{I}_1 & 0 & \overline{I}_2 \\ 0 & -\overline{I}_1 & 0 \\ \overline{I}_2 & 0 & \overline{I}_3 \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} = 0 \tag{25}$$

where

$$C_{11} = -\alpha_m^2 A_{11} - A_{55}/R^2,\ C_{22} = \alpha_m^2 A_{55} + A_{11}/R^2,\ C_{33} = -\alpha_m^2 D_{11} - A_{55}$$
$$C_{32} = -C_{23} = \alpha_m \left[\left(\frac{B_{11}}{R}\right) - A_{55}\right] C_{12} = -C_{21} = \alpha_m \left[\left(\frac{A_{11} + A_{55}}{R}\right)\right] \tag{26}$$
$$C_{31} = C_{13} = -\alpha_m^2 B_{11} + A_{55}/R^2$$

The nontrivial solution for natural frequencies can be found by setting the determinant of characteristic equation matrix to zero.

For boundary conditions other than simply supported, the General Differential Quadrature (GDQ) method is used to solve the equations. The method has been used by several investigators [13, 36, 37]. In this method, the derivative of a function is approximated by a weighted linear sum of the function values at all the discrete points.

$$\frac{dF^{(n)}(x_i)}{dx} = \sum_{j=1}^{N} C_{ij}^{(n)} F\left(x_j\right) \tag{27}$$

Generally, by considering the Lagrange interpolation polynomials as test functions, the coefficients for the first order derivative become

$$C_{ij,i\neq j}^{(1)} = \frac{M(x_i)}{(x_i - x_j) M(x_j)} \tag{28}$$

$$M(x_i) = \prod_{j=1, j\neq i}^{N} (x_i - x_j) \tag{29}$$

for higher order derivatives

$$C_{ij,i\neq j}^{(n)} = n\left(C_{ij}^{(1)}.C_{ii}^{(n-1)} - C_{ij}^{(n-1)} / \left(x_i - x_j\right)\right) \tag{30}$$

and for all derivatives

$$C_{ii}^{(n)} = -\sum_{j=1,\, j\neq i}^{N} C_{ij}^{(n)} \tag{31}$$

The sampling points are selected based on the Chebyshev–Gauss–Lobatto (C–G–L) grid distribution to get denser population near boundaries.

$$x_i = \frac{a}{2}\left[1-\cos\left(\frac{\pi(i-1)}{N-1}\right)\right] \tag{32}$$

Application of this method to the equations of motion with respect to displacement functions will give a system of equations in matrix form that in this chapter are solved using MATLAB®. Then, the results for static and free vibration are compared with those obtained using a 3D finite element model for different laminates (unidirectional, symmetric and asymmetric cross ply and symmetric and asymmetric angle-ply) in the next section.

Numerical results

A rectangular cross section beam model having 1 m length, 0.025 m width, and 0.05 m height was considered and modeled in ANSYS® finite element code (Fig. 4.2). Solid elements were used as the representative of 3D elasticity analysis. A convergence study was done and the converged model had 8 elements in the thickness direction, 4 elements in the width direction and 160 elements in the length direction. A ratio of length to height of 20 was selected (often considered as the limit for applying classical beam theory). The results of a simply supported beam for a/h=10 has also been derived and presented in Appendix A. The material properties of graphite/epoxy are E_1 = 138 GPa, E_2 =8.96 GPa, υ_{12}=0.3, G_{12}= 7.1 GPa, ρ =1580 kg/m^3.

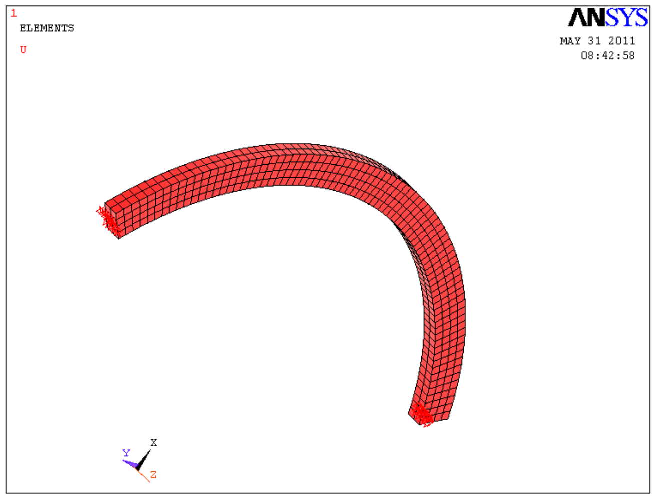

Figure 4.2 3D Finite Element Model of curved beam in ANSYS

Both static and modal analyses are done and the results are compared to those obtained using the proposed set of equations in order to check the accuracy of the model. A convergence study was also done for the GDQ analysis and it was found that using 11 point in the beam would yield convergent results for the first 3 significant figures.

Static analysis

Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_n a^4\right)$ and moment $\left(10M\right)/\left(p_n a^3\right)$ of simply supported, clamped-clamped and cantilever curved beams using the proposed method and 3D FEM are presented in the Tables 4.1 through 4.3. The beam is under uniform normal load p_n. In each case, the lower number shows the 3D FEA result. The moment here is determined by using equation 10. The moment using 3D FEA is determined by

$$M = \iint_A \sigma z dA \tag{33}$$

Since the results are very close for different deepness ratios of simply supported and cantilever boundary conditions, they are presented at fewer number of deepness ratios. The results show good accuracy for the proposed equations in both shallow and deep beams with various kinds of laminates. Some error in the results for the moment is due to inaccuracy in integration of the stress over the area especially in bending-twisting coupled laminates.

The first observation made here from Table 4.1 is that almost all results were within 5% error when compared with 3D elasticity. The accuracy of deformation prediction was higher than moment prediction for almost all cases considered here with the exception of the asymmetric angle ply laminate. In general, these observations are made for the cases of Tables 4.2 and 4.3.

Free vibration analysis

Tables 4.4 through 4.6 show the first five nondimensional natural frequencies $\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}$ of curved beams with different deepness and boundary conditions. The results for a simply supported case are compared to classic beam theory (CBT) [8] and FSDT with different stiffness parameters. As expected, one can see that the classic beam model using normal ABD parameters is only valid for 1st mode of cross-ply laminates, since it only takes B_{11} into account. The effective length (length of the half sine wave of the mode) becomes less for higher modes and the thin beam assumption leads to inaccurate results. Although the $[45]_4$ laminate is symmetric, it has bending twisting coupling and using the normal ABD formulation leads to inaccurate results.

Using FSDT approach for thick beams along the equivalent modulus of elasticity for calculation of the ABD parameters (Eqs. 5-7) one can reach accurate results for all cases. This approach does not have coupling problems and accurate results for all laminates are achieved. The overall range of error is about 1 percent. The results of GDQ analysis for other boundary conditions show the same behavior. These results can be used as a benchmark for analysis of curved beams.

Table 4.1 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_n a^4\right)$ and moment $\left(10M\right)/\left(p_n a^3\right)$ of simply supported curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

a/R	Method	**$[0]_4$**		**$[0/90]_s$**		**$[0_2/90_2]$**		**$[45]_4$**		**$[30_2/60_2]$**	
		w	M	w	M	w	M	w	M	w	M
0.2	Exact	1.752	3.140	1.963	3.140	7.203	3.140	14.16	3.140	13.21	3.140
	3D FEA	1.745	3.180	1.949	3.058	7.180	3.244	13.63	3.315	12.64	3.136
0.6	Exact	1.871	3.248	2.096	3.248	7.621	3.248	15.12	3.248	14.06	3.248
	3D FEA	1.869	3.292	2.087	3.169	7.614	3.362	14.57	3.430	13.47	3.220
1	Exact	2.149	3.489	2.407	3.489	8.671	3.489	17.37	3.489	16.10	3.489
	3D FEA	2.159	3.540	2.408	3.351	8.694	3.580	16.76	3.685	15.45	3.558
2	Exact	4.883	5.319	5.469	5.319	19.27	5.319	39.53	5.319	36.35	5.319
	3D FEA	5.030	5.423	5.593	5.084	19.58	5.633	38.36	5.627	35.14	5.368

Table 4.2 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_n a^4\right)$ and moment $\left(10M\right)/\left(p_n a^3\right)$ of clamped-clamped curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

a/R	Method	$[0]_4$		$[0/90]_s$		$[0_2/90_2]$		$[45]_4$		$[30_2/60_2]$	
		w	M	w	M	w	M	w	M	w	M
0.2	GDQ	0.3248	1.462	0.4053	1.674	0.9019	1.788	2.263	1.623	2.048	1.730
	3D FEA	0.3241	1.479	0.4005	1.722	0.9025	1.812	2.229	1.493	1.987	1.652
0.4	GDQ	0.1715	0.7712	0.2551	1.053	0.4016	1.049	1.362	0.9752	1.177	1.087
	3D FEA	0.1720	0.7960	0.2539	1.092	0.3980	1.029	1.345	0.9004	1.138	1.023
0.6	GDQ	0.09603	0.4316	0.1578	0.6508	0.2090	0.6769	0.8193	0.5848	0.6885	0.6892
	3D FEA	0.09670	0.4542	0.1578	0.6670	0.2059	0.6461	0.8109	0.5402	0.6646	0.6243
0.8	GDQ	0.05948	0.2670	0.1030	0.4239	0.1253	0.4835	0.5261	0.3739	0.4355	0.4694
	3D FEA	0.06010	0.2747	0.1034	0.4379	0.1228	0.4658	0.5217	0.3424	0.4203	0.4291
1	GDQ	0.03998	0.1792	0.07122	0.2927	0.08284	0.3706	0.3605	0.2549	0.2958	0.3413
	3D FEA	0.04053	0.1846	0.07172	0.2992	0.08090	0.3527	0.3581	0.2339	0.2855	0.3099
2	GDQ	0.01084	0.04800	0.02017	0.08158	0.02201	0.1637	0.1001	0.06771	0.08074	0.1226
	3D FEA	0.01123	0.05121	0.02060	0.07982	0.02119	0.1545	0.1005	0.06203	0.07840	0.1084

Table 4.3 Nondimensional maximum deflection $\left(10^3 E_1 h^3 w\right)/\left(p_n a^4\right)$ and moment $\left(10M\right)/\left(p_n a^3\right)$ of clamped-free curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

a/R	Method	$[0]_4$		$[0/90]_s$		$[0_2/90_2]$		$[45]_4$		$[30_2/60_2]$	
		w	M	w	M	w	M	w	M	w	M
0.2	GDQ	15.49	12.46	17.47	12.46	66.63	12.46	131.9	12.46	123.0	12.46
	3D FEA	15.49	12.23	17.41	11.89	66.64	12.83	131.0	10.99	120.9	11.69
0.6	GDQ	14.65	12.13	16.52	12.13	62.52	12.13	124.9	12.13	116.1	12.13
	3D FEA	14.68	11.64	16.51	11.86	62.63	11.88	124.1	10.71	114.3	11.45
1	GDQ	13.10	11.49	14.78	11.49	55.52	11.49	112.1	11.49	103.9	11.49
	3D FEA	13.18	11.26	14.81	11.25	55.74	11.75	111.3	10.15	102.2	10.97
2	GDQ	7.788	8.851	8.770	8.851	32.66	8.851	66.28	8.851	61.19	8.851
	3D FEA	7.832	8.743	8.789	8.582	32.37	8.875	65.66	7.921	59.93	8.321

Table 4.4 Nondimensional natural frequencies ($\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}$) of simply supported curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, $\rho = 1580\,\mathrm{kg/m^3}$)

$[0]_4$												
	a/R=0.2				**a/R=0.4**				**a/R=0.6**			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	9.839	9.375	9.350	9.300	9.660	9.205	9.181	9.129	9.368	8.928	8.904	8.849
2	39.53	33.36	33.29	32.76	39.35	33.21	33.14	32.60	39.06	32.96	32.89	32.34
3	89.03	64.49	64.39	62.70	88.85	64.36	64.26	62.56	88.55	64.14	64.04	62.33
4	158.3	98.07	97.96	94.61	158.1	97.96	97.85	94.49	157.8	97.78	97.67	94.30
5	247.4	132.2	132.1	128.3	247.2	132.1	132.0	126.7	246.9	131.9	131.8	126.6
	a/R=0.8				**a/R=1**				**a/R=2**			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	8.970	8.550	8.527	8.467	8.477	8.080	8.058	7.994	4.967	4.737	4.725	4.647
2	82.96	70.02	69.87	69.01	81.82	69.07	68.92	68.02	72.80	61.49	61.36	60.18
3	189.2	137.1	136.8	134.3	188.0	136.2	136.0	133.4	178.7	129.5	129.3	126.3
4	338.0	209.4	209.1	204.3	336.8	208.7	208.4	203.6	327.3	202.8	202.6	197.3
5	529.2	282.7	282.5	275.3	528.0	282.1	281.9	274.6	518.4	277.0	276.8	269.2
$[0/90]_s$												
	a/R=0.2				**a/R=0.4**				**a/R=0.6**			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	9.246	8.857	8.833	8.801	9.078	8.696	8.673	8.639	8.804	8.434	8.412	8.375
2	37.15	31.88	31.81	31.40	36.98	31.74	31.67	31.26	36.70	31.50	31.43	31.01
3	83.66	62.33	62.23	60.80	83.49	62.20	62.10	60.67	83.21	62.00	61.89	60.45
4	148.8	95.63	95.51	92.47	148.6	95.52	95.40	92.36	148.3	95.34	95.22	92.18
5	232.5	129.7	129.6	126.1	232.3	129.6	129.5	126.4	232.0	129.4	129.3	124.3
	a/R=0.8				**a/R=1**				**a/R=2**			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	8.430	8.077	8.055	8.015	7.966	7.633	7.613	7.569	4.668	4.476	4.464	4.407
2	36.31	31.17	31.10	30.67	35.81	30.74	30.68	30.23	31.86	27.37	27.31	26.78
3	82.81	61.70	61.60	60.15	82.31	61.33	61.23	59.77	78.19	58.30	58.20	56.62
4	147.9	95.09	94.97	91.91	147.4	94.76	94.64	91.58	143.2	92.10	91.99	88.83
5	231.6	129.2	129.1	124.1	231.1	128.9	128.8	123.8	226.9	126.6	126.5	121.4

Table 4.4 (continued)

$[0_2/90_2]$												
	a/R=0.2				a/R=0.4				a/R=0.6			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	4.638	4.584	4.611	4.596	4.534	4.481	4.548	4.531	4.379	4.329	4.430	4.412
2	18.61	17.78	17.88	17.64	18.44	17.63	17.87	17.63	18.22	17.42	17.81	17.56
3	41.77	37.97	38.16	37.16	41.50	37.75	38.23	37.21	41.18	37.49	38.24	37.21
4	73.98	63.28	63.56	61.02	73.56	63.00	63.70	61.14	73.10	62.68	63.79	61.20
5	115.0	92.07	92.42	87.58	114.4	91.74	92.63	87.74	113.8	91.38	92.79	87.87
	a/R=0.8				a/R=1				a/R=2			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	4.177	4.130	4.259	4.239	3.933	3.890	4.040	4.019	2.278	2.254	2.399	2.377
2	17.95	17.17	17.70	17.44	17.63	16.88	17.53	17.27	15.40	14.78	15.91	15.61
3	40.81	37.18	38.21	37.16	40.38	36.83	38.13	37.06	37.61	34.45	36.93	35.78
4	72.59	62.32	63.84	61.23	72.03	61.92	63.84	61.20	68.58	59.31	63.13	60.35
5	113.1	90.97	92.90	87.95	112.3	90.53	92.98	87.98	108.1	87.74	92.69	88.33
$[45]_4$												
	a/R=0.2				a/R=0.4				a/R=0.6			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	5.627	5.532	3.283	3.331	5.525	5.432	3.224	3.271	5.358	5.269	3.127	3.172
2	22.61	21.20	12.94	12.94	22.51	21.11	12.89	12.88	22.34	20.95	12.79	12.80
3	50.92	44.53	28.29	28.24	50.82	44.44	28.23	28.18	50.64	44.30	28.14	28.10
4	90.55	72.90	48.39	48.13	90.45	72.82	48.34	48.08	90.27	72.68	48.25	48.00
5	141.5	104.2	72.30	71.54	141.4	104.2	72.25	71.49	141.2	104.0	72.17	71.41
	a/R=0.8				a/R=1				a/R=2			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	5.131	5.046	2.995	3.038	4.848	4.769	2.830	2.871	2.841	2.796	1.659	1.682
2	22.10	20.73	12.65	12.67	21.80	20.45	12.48	12.51	19.39	18.21	11.12	11.17
3	50.40	44.09	28.01	27.97	50.10	43.83	27.84	27.82	47.60	41.68	26.48	26.52
4	90.03	72.49	48.13	47.88	89.72	72.25	47.97	47.72	87.19	70.26	46.67	46.48
5	141.0	103.9	72.05	71.30	140.7	103.6	71.90	71.16	138.1	101.8	70.67	70.00
$[30_2/60_2]$												
	a/R=0.2				a/R=0.4				a/R=0.6			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	5.127	5.054	3.402	3.463	5.048	4.976	3.347	3.406	4.909	4.839	3.251	3.308
2	20.58	19.49	13.39	13.43	20.55	19.45	13.35	13.40	20.45	19.36	13.28	13.33
3	46.28	41.30	29.20	29.34	46.32	41.31	29.19	29.26	46.30	41.26	29.14	29.18
4	82.13	68.23	49.82	49.68	82.27	68.29	49.84	49.69	82.36	68.30	49.83	49.66
5	128.0	98.42	74.23	73.58	128.3	98.52	74.28	73.61	128.5	98.57	74.30	73.62
	a/R=0.8				a/R=1				a/R=2			
n	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM	CBT	FSDT (Q_{11})	FSDT (E_x)	3D FEM
1	4.712	4.645	3.119	3.172	4.463	4.400	2.951	3.002	2.634	2.598	1.738	1.765
2	20.29	19.20	13.16	13.22	20.07	18.99	13.00	13.07	18.08	17.10	11.67	11.75
3	46.21	41.17	29.05	29.08	46.06	41.01	28.93	28.96	44.37	39.43	27.73	27.83
4	82.38	68.25	49.78	49.60	82.33	68.16	49.69	49.50	81.16	66.93	48.70	48.52
5	128.6	98.57	74.28	73.59	128.7	98.53	74.23	73.53	128.3	97.61	73.46	72.74

Table 4.5 Nondimensional natural frequencies $\left(\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}\right)$ of clamped-clamped curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, $\rho = 1580\,\text{kg/m}^3$)

$[0]_4$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	21.72	21.76	29.58	29.60	38.67	38.68
2	42.42	42.70	42.17	42.45	41.78	42.04
3	71.69	72.40	72.23	72.92	73.45	74.08
4	102.8	104.1	102.5	103.9	102.2	103.5
5	135.7	137.4	135.7	137.4	135.3	137.4
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	47.14	47.21	53.54	53.74	NA	NA
2	41.23	41.48	40.56	40.79	35.73	35.83
3	75.96	76.49	80.71	81.07	59.32	59.72
4	101.7	102.9	101.0	102.3	96.56	97.58
5	135.4	137.5	136.0	137.6	148.5	149.0
$[0/90]_s$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	19.49	19.62	24.42	24.52	30.70	30.79
2	41.13	41.70	40.89	41.45	40.50	41.05
3	69.91	71.16	70.14	71.37	70.60	71.80
4	100.8	103.0	100.6	102.7	100.1	102.2
5	133.6	135.9	133.2	136.3	133.1	136.3
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	37.32	37.42	43.54	43.70	NA	NA
2	39.97	40.50	39.31	39.82	34.61	34.97
3	71.47	72.62	73.04	74.08	56.07	56.78
4	99.57	101.6	98.89	100.9	94.22	95.96
5	133.0	136.1	132.9	136.0	134.5	136.7
$[0_2/90_2]$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	13.07	13.09	19.32	19.47	26.04	26.37
2	25.90	25.75	25.66	25.70	25.35	25.57
3	47.62	47.01	47.94	47.62	48.86	48.84
4	72.80	71.55	72.42	71.62	71.94	71.58
5	101.0	98.80	100.7	99.07	100.5	99.37

Table 4.5 (continued)

[$0_2/90_2$]						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	31.88	32.41	35.85	36.60	NA	NA
2	24.96	25.35	24.50	25.04	21.60	22.45
3	50.96	51.24	54.92	55.55	38.53	40.27
4	71.38	71.44	70.76	71.22	66.84	68.96
5	100.4	99.71	100.3	100.1	107.7	109.8
[$45]_4$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	8.285	8.350	10.63	10.72	13.60	13.69
2	19.49	19.64	19.38	19.53	19.21	19.36
3	36.74	37.01	36.81	37.08	36.96	37.23
4	57.83	58.19	57.62	58.00	57.31	57.70
5	82.20	82.60	82.12	82.56	82.05	82.50
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	16.76	16.87	19.86	19.98	28.61	28.78
2	18.98	19.12	18.69	18.83	16.59	16.72
3	37.26	37.50	37.69	37.96	46.68	46.99
4	56.92	57.31	56.42	56.85	53.63	54.10
5	81.96	82.42	81.86	82.32	81.31	81.83
[$30_2/60_2$]						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	8.709	8.842	11.44	11.49	14.82	15.11
2	20.11	20.36	20.04	20.28	19.90	20.14
3	37.81	38.29	37.97	38.43	38.22	38.68
4	59.36	59.96	59.28	59.86	59.09	59.67
5	84.10	84.82	84.17	84.88	84.22	84.92
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	18.36	18.72	21.77	22.19	30.36	30.72
2	19.69	19.93	19.41	19.65	17.33	17.53
3	38.64	39.11	39.32	39.82	50.96	51.97
4	58.81	59.38	58.46	59.02	55.93	56.54
5	84.25	84.95	84.27	84.96	84.46	85.22

Table 4.6 Nondimensional natural frequencies $\left(\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}\right)$ of cantilever curved beam (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, $\rho = 1580\,\text{kg/m}^3$)

$[0]_4$								
	a/R=0.2		**a/R=0.6**		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	3.438	3.439	3.460	3.458	3.506	3.498	3.730	3.687
2	18.99	19.32	18.28	18.31	17.11	17.12	13.95	13.87
3	46.20	46.40	45.44	45.62	44.16	44.30	39.92	39.85
4	77.72	78.27	76.89	77.40	75.66	76.11	71.86	72.05
5	112.0	112.9	108.8	109.6	108.2	109.1	105.5	106.1
$[0/90]_s$								
	a/R=0.2		**a/R=0.6**		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	3.239	3.245	3.261	3.263	3.304	3.302	3.513	3.485
2	18.11	18.21	17.43	17.52	16.31	16.39	13.31	13.30
3	44.51	44.98	43.71	44.16	42.43	42.81	38.33	38.53
4	75.27	76.29	73.90	74.84	72.70	73.62	69.37	70.09
5	108.9	110.8	109.0	110.9	109.6	111.3	101.4	102.7
$[0_2/90_2]$								
	a/R=0.2		**a/R=0.6**		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	1.660	1.659	1.674	1.673	1.700	1.697	1.817	1.807
2	10.05	10.00	9.807	9.751	9.294	9.231	7.714	7.633
3	26.74	26.46	26.48	26.19	25.93	25.62	23.89	23.54
4	49.28	48.45	49.25	48.40	48.85	47.99	47.12	46.19
5	75.00	73.52	74.38	72.94	72.51	72.66	73.50	71.78
$[45]_4$								
	a/R=0.2		**a/R=0.6**		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	1.182	1.191	1.190	1.199	1.205	1.216	1.280	1.299
2	7.233	7.280	6.978	7.024	6.555	6.598	5.385	5.424
3	19.74	19.86	19.41	19.53	18.87	19.01	17.12	17.03
4	36.47	36.70	36.00	36.21	35.57	35.81	34.14	34.47
5	59.06	59.29	59.03	59.26	59.03	59.28	52.06	52.46
$[30_2/60_2]$								
	a/R=0.2		**a/R=0.6**		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	1.223	1.238	1.232	1.247	1.248	1.266	1.328	1.354
2	7.492	7.576	7.265	7.345	6.849	6.924	5.648	5.710
3	20.38	20.59	20.09	20.29	19.58	19.78	17.88	17.95
4	38.40	38.73	37.86	38.23	37.30	37.69	35.81	36.33
5	60.69	61.15	60.92	61.37	61.27	61.75	55.26	56.06

The mode shapes for the first five modes of simply supported curved beam with a/R=1 and laminate $[30_2/60_2]$ are presented in the following figures. One should note the twist induced in the mode shapes due to bending-twisting coupling.

Figure 4.3 Mode shape number 1 of curved beam with $[30_2/60_2]$ laminate

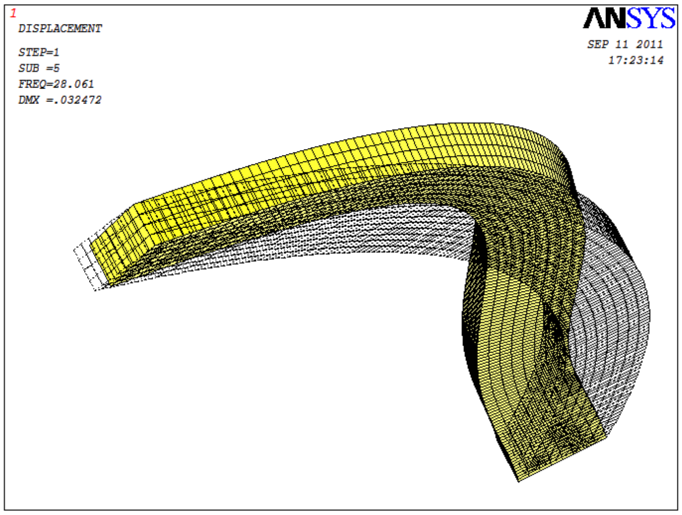

Figure 4.4 Mode shape number 2 of curved beam with $[30_2/60_2]$ laminate

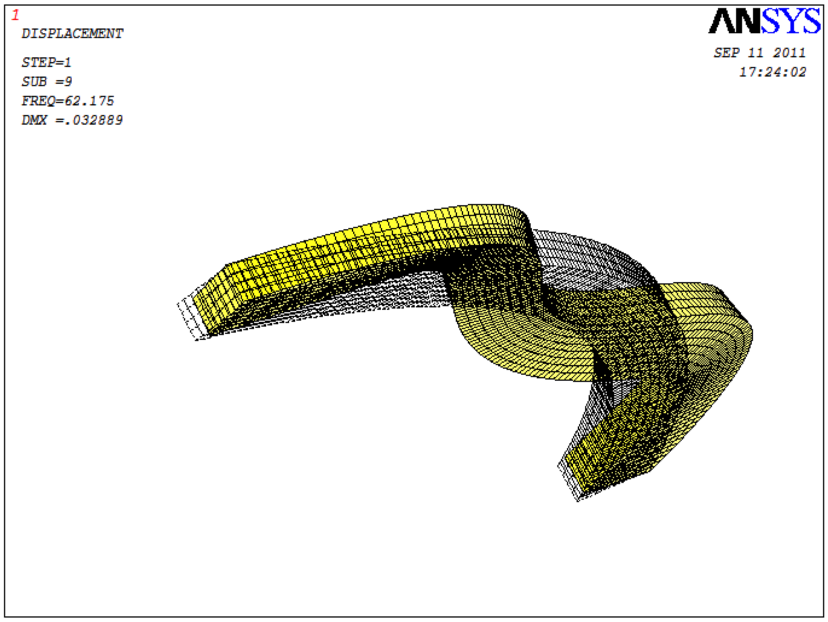

Figure 4.5 Mode shape number 3 of curved beam with $[30_2/60_2]$ laminate

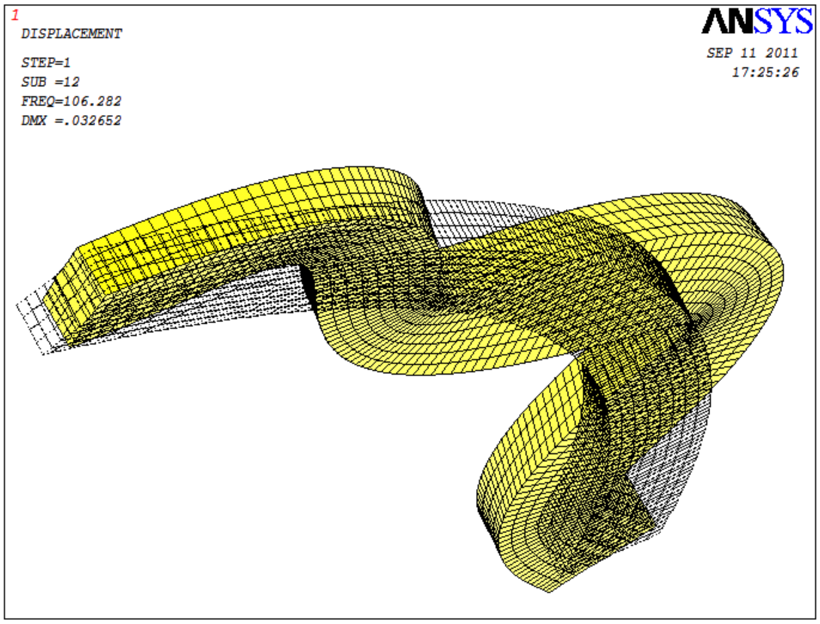

Figure 4.6 Mode shape number 4 of curved beam with $[30_2/60_2]$ laminate

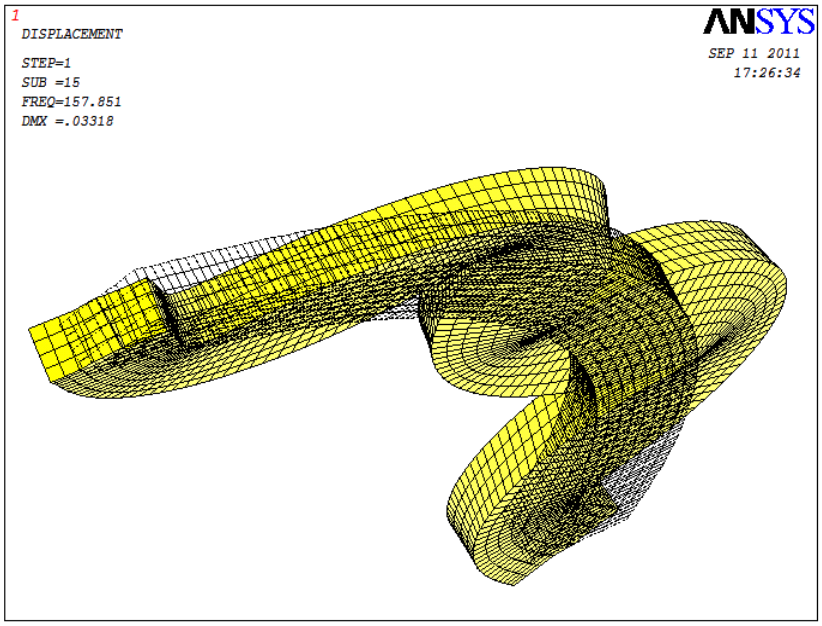

Figure 4.7 Mode shape number 5 of curved beam with [$30_2/60_2$] laminate

Comparison to higher order shear deformation theory

In order to check the accuracy of higher order shear deformation theories, the third order theory by Reddy [6] using modified ABDs is used. The procedure is very similar to FSDT. Hence, only formulations are proposed here.

Kinematic Relations:

$$u = \left(1 + \frac{z}{R}\right)u_0 + z\psi + cz^3\left(\psi + \frac{\partial w}{\partial \alpha}\right) \quad c = \frac{4}{3h^2} \tag{34}$$

$$w = w_0 \tag{35}$$

$$\varepsilon = \varepsilon_0 + z\varepsilon_1 + z^3\varepsilon_2 \tag{36}$$

$$\gamma = \gamma_0 + z^2\gamma_1 \tag{37}$$

$$\varepsilon_0 = \frac{\partial u_0}{\partial \alpha} + \frac{w_0}{R} \tag{38}$$

$$\varepsilon_1 = \frac{\partial \psi}{\partial \alpha} \tag{39}$$

$$\varepsilon_2 = -c\left(\frac{\partial \psi}{\partial \alpha} + \frac{\partial^2 w}{\partial \alpha^2} - \frac{1}{R}\frac{\partial u}{\partial \alpha}\right) \tag{40}$$

$$\gamma_0 = \left(\psi + \frac{\partial w}{\partial \alpha} - \frac{u}{R}\right) \tag{41}$$

$$\gamma_1 = -3c\left(\psi + \frac{\partial w}{\partial \alpha} - \frac{u}{R}\right) \tag{42}$$

Equations of Motion:

$$\frac{\partial N}{\partial \alpha} + \frac{Q}{R} = I_0 \ddot{u}_0 + J_1 \ddot{\psi} - cI_3 \frac{\partial \ddot{w}_0}{\partial \alpha} \tag{43}$$

$$\frac{\partial Q}{\partial \alpha} - \frac{N}{R} = I_0 \ddot{w}_0 - 9c^2 I_6 \frac{\partial^2 \ddot{w}_0}{\partial \alpha^2} + 3cI_3 \frac{\partial \ddot{u}_0}{\partial \alpha} + J_4 \frac{\partial \ddot{\psi}}{\partial \alpha} \tag{44}$$

$$\frac{\partial M}{\partial \alpha} - Q = J_1 \ddot{u}_0 + K_2 \ddot{\psi} - 3cJ_4 \frac{\partial \ddot{w}_0}{\partial \alpha} \tag{45}$$

$$I_i = \int_A \rho\left(1 + \frac{z}{R}\right) z^i dz \tag{46}$$

$$J_i = I_i - cI_{i+2} \tag{47}$$

$$K_2 = I_2 - 2cI_4 + c^2 I_6 \tag{48}$$

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A & B & E & 0 & 0 \\ B & D & F & 0 & 0 \\ 0 & 0 & 0 & A & D \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \gamma_0 \\ \gamma_1 \end{bmatrix} \tag{49}$$

$$A_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \ln\left(\frac{R+h_k}{R+h_{k-1}}\right) \tag{50}$$

$$B_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \left[\left(h_k - h_{k-1}\right) - R\ln\left(\frac{R+h_k}{R+h_{k-1}}\right)\right] \tag{51}$$

$$D_{11} = R\sum_{k=1}^{N} bE_x^{(k)} \left[\frac{1}{2}\left(\left(R+h_k\right)^2 - \left(R+h_{k-1}\right)^2\right) - 2R\left(h_k - h_{k-1}\right) - R^2 \ln\left(\frac{R+h_k}{R+h_{k-1}}\right)\right] \tag{52}$$

$$E_{11} = \sum_{k=1}^{N} bE_x^{(k)} \left[\frac{\left(h_k^4 - h_{k-1}^4\right)}{4} - \frac{\left(h_k^5 - h_{k-1}^5\right)}{5R} + \frac{\left(h_k^6 - h_{k-1}^6\right)}{6R^2}\right] \tag{53}$$

$$F_{11} = \sum_{k=1}^{N} bE_x^{(k)} \left[\frac{\left(h_k^5 - h_{k-1}^5\right)}{5} - \frac{\left(h_k^6 - h_{k-1}^6\right)}{6R} + \frac{\left(h_k^7 - h_{k-1}^7\right)}{7R^2}\right] \tag{54}$$

The formulation for the first three terms is based on exact integration of deepness term. However, the last two terms are derived by expansion of the deepness term since an exact integration gives a long expression. Expressing the equations of motion in matrix form, gives the following terms of equation 18.

$$L_{11} = \left(A_{11} + \frac{E_{11}c}{R}\right)\frac{\partial^2}{\partial\alpha^2} - \frac{A_{55}}{R^2} + \frac{3cD_{55}}{R^2}, L_{12} = \left(\frac{A_{11} + A_{55} - 3cD_{55}}{R}\right)\frac{\partial}{\partial\alpha} - E_{11}c\frac{\partial^3}{\partial\alpha^3}$$

$$L_{13} = \left(B_{11} - cE_{11}\right)\frac{\partial^2}{\partial\alpha^2} + \frac{A_{55} - 3cD_{55}}{R},\ L_{21} = \left(\frac{-A_{11} - A_{55} + 3cD_{55}}{R} - \frac{E_{11}c}{R^2}\right)\frac{\partial}{\partial\alpha}$$

$$L_{22} = \left(A_{55} - 3cD_{55} + \frac{cE_{11}}{R}\right)\frac{\partial^2}{\partial\alpha^2} - \frac{A_{11}}{R^2},\ L_{23} = \left(\frac{-B_{11} + E_{11}c}{R} + A_{55} - 3cD_{55}\right)\frac{\partial}{\partial\alpha}$$

$$L_{31} = \left(B_{11} + \frac{cF_{11}}{R} \right) \frac{\partial^2}{\partial \alpha^2} + \frac{A_{55} - 3cD_{55}}{R}, \; L_{32} = \left(\frac{B_{11}}{R} - A_{55} + 3cD_{55} \right) \frac{\partial}{\partial \alpha} - cF_{11} \frac{\partial^3}{\partial \alpha^3}$$

$$L_{33} = \left(D_{11} - cF_{11} \right) \frac{\partial^2}{\partial \alpha^2} - A_{55} + 3cD_{55}$$

By applying displacement functions for simply supported boundary conditions we have the following terms in equation 25.

$$C_{11} = -\alpha^2 \left(A_{11} + \frac{E_{11}c}{R} \right) - \frac{A_{55} - 3cD_{55}}{R^2}, \; C_{12} = \alpha \left(\frac{A_{11} + A_{55} - 3cD_{55}}{R} \right) - \alpha^3 cE_{11}$$

$$C_{13} = -\alpha^2 \left(B_{11} - cE_{11} \right) + \frac{A_{55} - 3cD_{55}}{R}, \; C_{21} = \alpha \left(\frac{A_{11} + A_{55} - 3cD_{55}}{R} + \frac{E_{11}c}{R^2} \right)$$

$$C_{22} = -\alpha^2 \left(A_{55} - 3cD_{55} + \frac{cE_{11}}{R} \right) - \frac{A_{11}}{R^2}, \; C_{23} = \alpha \left(\frac{B_{11} - E_{11}c}{R} - A_{55} + 3cD_{55} \right)$$

$$C_{31} = -\alpha^2 \left(B_{11} + \frac{cF_{11}}{R} \right) + \frac{A_{55} - 3cD_{55}}{R}, \; C_{32} = \alpha \left(\frac{B_{11}}{R} - A_{55} + 3cD_{55} \right) + \alpha^3 cF_{11}$$

$$C_{33} = -\alpha^2 \left(D_{11} - cF_{11} \right) - A_{55} + 3cD_{55}$$

Then, making determinant of equation 25 equal to zero gives the natural frequencies. The HSDT capability in prediction of natural frequencies for nearly straight (a/R=0.1) and nearly deep (a/R=1) composite beams has been compared to present FSDT. The results for the first five natural frequencies are presented in table 4.7.

Table 4.7 Comparison of nondimensional natural frequencies ($\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}$) using $(ABD)_e$ by FSDT and HSDT for shallow and deep beams (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, $\rho = 1580\,kg/m^3$)

	a/R=0.1			a/R=1		
$[0]_4$						
n	FSDT (E_x)	HSDT (E_x)	3D FEM	FSDT (E_x)	HSDT (E_x)	3D FEM
1	9.392	9.309	9.358	8.058	7.987	8.006
2	33.33	32.43	32.96	32.10	31.23	31.68
3	64.42	61.58	63.27	63.36	60.57	62.15
4	97.99	92.46	95.79	97.08	91.60	94.81
5	132.1	123.5	128.7	131.3	122.8	127.9
$[0/90]_s$						
n	FSDT (E_x)	HSDT (E_x)	3D FEM	FSDT (E_x)	HSDT (E_x)	3D FEM
1	8.873	8.813	8.859	7.613	7.562	7.584
2	31.85	31.17	31.63	30.68	30.02	30.41
3	62.26	60.05	61.47	61.23	59.06	60.39
4	95.53	91.10	93.88	94.64	90.25	92.94
5	129.6	122.6	126.9	128.8	121.8	126.1
$[0_2/90_2]$						
n	FSDT (E_x)	HSDT (E_x)	3D FEM	FSDT (E_x)	HSDT (E_x)	3D FEM
1	4.622	6.324	4.611	4.040	NA	4.022
2	17.86	22.23	17.66	17.53	9.00	17.30
3	38.10	46.37	37.25	38.13	35.36	37.21
4	63.47	76.49	61.32	63.84	65.52	61.58
5	92.31	110.7	88.21	92.98	99.17	88.7
$[45]_4$						
n	FSDT (E_x)	HSDT (E_x)	3D FEM	FSDT (E_x)	HSDT (E_x)	3D FEM
1	3.298	3.295	3.349	2.830	2.827	2.873
2	12.96	12.90	12.96	12.48	12.43	12.52
3	28.30	28.05	28.30	27.84	27.59	27.87
4	48.41	47.70	48.29	47.97	47.27	47.87
5	72.31	70.80	71.87	71.90	70.40	71.48
$[30_2/60_2]$						
n	FSDT (E_x)	HSDT (E_x)	3D FEM	FSDT (E_x)	HSDT (E_x)	3D FEM
1	3.414	3.326	3.481	2.951	NA	3.007
2	13.39	13.50	13.46	13.00	11.45	13.09
3	29.19	29.47	29.26	28.93	27.58	29.04
4	49.80	50.11	49.85	49.69	48.31	49.70
5	74.19	74.30	73.95	74.23	72.55	73.93

The results show that the modified HSDT can accurately predict natural frequencies of shallow beams and symmetric deep beams. However, this method underestimates the natural frequencies of unsymmetric deep beams and even does not give result for the first natural frequency.

Conclusion

A modified FSDT model that accounts for deepness, laminate couplings, shear deformation and rotary inertia was validated for static and free vibration analysis of composite curved beams. The proposed model uses deep formulation along with lamina modulus for calculation of ABD parameters. The method was verified using 3D FEM model. The results showed good accuracy of the model for rectangular beams in static and vibration analyses for all kinds of laminates. The model has also been compared to HSDT and it is shown that FSDT using modified ABDs provides an accurate set of equations for calculating the deflection, stress, and natural frequencies of straight and curved beams with arbitrary laminate.

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CHAPTER V

TRANSVERSE VIBRATION ANALYSIS OF GENERALLY LAMINATED MULTI-SEGMENT COMPOSITE SHAFTS WITH A LUMPED MASS

Introduction

Higher strength and stiffness to weight ratios along with the ability to tailor the design for specific purposes, have led to the extensive use of composite materials. Their weight savings in different industries such as aerospace or automotive has been of great interest because of its general positive impact on fuel economy, durability as well as other attributes including noise, vibration and harshness (NVH). In recent years, metal shafts are being targeted for possible replacement with composite ones in many applications such as helicopters, automotive vehicles and centrifugal separators. Considerable research has been dedicated to vibration analysis of shafts using beam and shell theories. Singh et al. [1] reviewed the developments in dynamics analysis of composite shafts in 1997. Kim et al. [2] investigated the free vibration of a rotating tapered composite Timoshenko shaft using Galerkin method.

Song et al. [3] addressed problems related to the implications of conservative and gyroscopic forces on vibration and stability of a circular cylindrical shaft modeled as a thin-walled spinning composite beam. Chang et al. [4] considered a composite shaft containing discrete isotropic rigid disks and supported by bearings modeled as springs and viscous dampers. They incorporated the transverse shear deformation, rotary inertia and gyroscopic effects, and validated their model for shafts with bending-twisting coupling. Chang et al. [5] performed vibration analysis of rotating composite shafts containing randomly oriented reinforcements. Gubran and Gupta [6] analyzed the natural frequencies of composite tubular shafts using equivalent modulus beam theory with shear deformation, rotary inertia and gyroscopic effects included. Their approach took into account different terms in the ABD matrix. Their results were compatible with those of layerwise theory [7]. Banerjee and Su [8] developed the dynamic stiffness matrix of a spinning thin-walled composite beam and investigated its free vibration characteristics based on the classic beam theory (CBT). They included bending twisting coupling effects. Na et al. [9] studied vibration and stability of a circular cylindrical shaft modeled as a tapered thin-walled composite beam, spinning with constant speed and subjected to an axial compressive force. Sino et al. [10] introduced a homogenized FEM which took into account internal damping of the beam and evaluated natural frequencies and instability thresholds of an internally damped rotating composite shaft. Qatu and Iqbal [11-12] studied transverse vibration of two-segment shafts using CBT. Their formulation was limited to long simply supported shafts with cross-ply laminates.

In this chapter, a shear deformation beam model that can accurately predict natural frequencies of generally laminated multi-segment shafts is presented. It is used here to obtain natural frequencies of two segment shafts and can be used by researchers and practicing engineers.

Complexities in analysis of laminated shafts

There are three complexities in the analysis of composite two-segment thick shafts: different segments, shear deformation, and material couplings. These effects have not been considered simultaneously in the literature. In this chapter, a simple method is proposed that considers all of these problems.

Effect of shear deformation

The inclusion of shear deformation in the analysis of beams was first made by Timoshenko [13] and this inclusion has been incorporated by many researchers. This effect is higher in composite materials because their longitudinal to shear modulus ratio is much higher than metallic materials. Consequently, the assumption of classical Euler-Bernoulli theory is not accurate. However, some researchers used this theory especially for getting exact solutions of long shafts [8,11-12]. In this chapter, a first order shear deformation theory (FSDT), where both shear deformation and rotary inertia are considered, is used.

Multi segment shafts

The driveshafts design is mostly dictated by its natural frequencies. In order to maximize the natural frequencies, shafts are built in one, two or more segments. Multi-segmented shafts or drivelines are usually analyzed as separate independent isolated segments, each of which is simply supported [14-15]. In such systems, the effect of one shaft segment bending stiffness on the other shaft cannot be ignored. This effect has a significant impact on the first natural bending frequency. This was observed for multi-segment metallic shafts [11-12].

Material coupling

The coupling complexity arises from the laminate sequence. In some applications, the structure is very weight limited or the laminate design is going to be tailored in order to get the required natural frequencies. In such applications, the laminate usually has one of the material couplings that should be taken into account in the analysis of the structure.

In typical beam models, 18 parameters in the ABD stiffness matrix are reduced to only 3 parameters (A_{11}, B_{11}, D_{11}) and only one term of the bending-stretching coupling is taken into account (i.e. B_{11}). If other coupling terms are not zero, there will be errors leading to inaccurate results. This complexity has been shown to be very important and introduces errors in the analysis [4,6,8,17]. This problem raises the need to redefine the stiffness parameters so that the couplings can be considered in these three stiffness terms. Some researchers defined other stiffness parameters for shafts [18-20].

Another problem that appears in asymmetric laminates is that in-plane and out-of-plane vibrations are coupled and those equations must be solved together. But for in-plane flexural vibrations, this has a very little effect [20]. However, for out-of-plane or more importantly extensional modes, this effect is high and the equations should be solved coupled together.

Hajianmaleki and Qatu [20] showed that using equivalent modulus of elasticity of each lamina, one can get accurate results for static and dynamic analyses of generally laminated beams with any kind of coupling. The equivalent modulus of elasticity of each lamina is found using equation 1.

$$\frac{1}{E_x^k}=\frac{\cos^4(\theta_k)}{E_{11}}+\left(\frac{1}{G_{12}}-\frac{2\upsilon_{12}}{E_{11}}\right)\cos^2(\theta_k)\sin^2(\theta_k)+\frac{\sin^4(\theta_k)}{E_{22}} \tag{1}$$

Figure 5.1 shows the cross sectional geometry of the shaft. Using modified stiffness parameter for the shaft, we have for ABD terms

$$\mathrm{A}_{11}=\pi\sum_{k=1}^{N}\mathrm{E}_x^{(k)}\left(r_k^2-r_{k-1}^2\right) \tag{2}$$

$$\mathrm{D}_{11}=\frac{\pi}{4}\sum_{k=1}^{N}\mathrm{E}_x^{(k)}\left(r_k^4-r_{k-1}^4\right) \tag{3}$$

where r is external radius of the each layer. The B_{11} formulation has been taken from research by Chan and Demirhan [19]

$$B_{11}=\pi R\sum_{k=1}^{n}\left[c^4Q_{11}+c^2\,s^2\left(Q_{12}+2Q_{66}\right)+\frac{3}{8}s^4\,Q_{22}\right]\left(z_k^2-z_{k-1}^2\right) \tag{4}$$

where $c=\cos(\theta), s=\sin(\theta)$, θ is the lamina angle relative to the shaft axis and z is the distance from middle layer through the thickness.

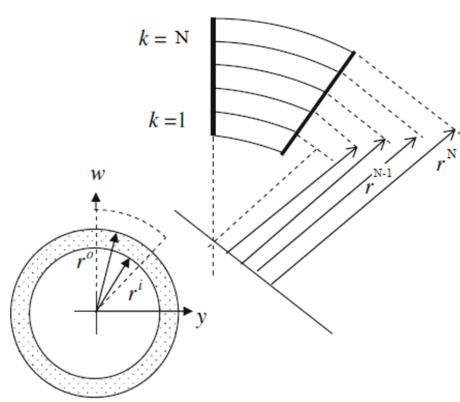

Figure 5.1 Cross sectional geometry of composite shaft

Formulation

Kinematic relations

In the FSDT formulation, the displacements, curvature changes and strains are

$$u = u_0 + z\psi, \quad w = w_0 \tag{5}$$

$$\varepsilon_0 = \frac{\partial u_0}{\partial x}, \qquad \kappa = \frac{\partial \psi}{\partial x}, \ \gamma = \frac{\partial w}{\partial x} + \psi \tag{6}$$

where u and w are displacements in x and z directions; respectively. ε_0 is middle surface strain, γ is the shear strain at the neutral axis and ψ is the rotation of a line element perpendicular to the original direction, respectively. Normal strain at any point is

$$\varepsilon = \varepsilon_0 + z\kappa \tag{7}$$

Force and moment resultants as well as shear forces are calculated using

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \kappa \\ \gamma \end{bmatrix} \tag{8}$$

where for A_{55} we have

$$A_{55} = k\overline{Q_{55}}A \tag{9}$$

where A is cross sectional area and k is shear correction factor that is assumed to be 0.5 for tubular cross sections.

Equations of motion

The equations of motion are

$$\frac{\partial N}{\partial x} = I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^2 \psi}{\partial t^2} - p_x \tag{10}$$

$$-\frac{\partial Q}{\partial x} = p_z - I_1 \frac{\partial^2 w}{\partial t^2} \tag{11}$$

$$\frac{\partial M}{\partial x} - Q = I_2 \frac{\partial^2 u}{\partial t^2} + I_3 \frac{\partial^2 \psi}{\partial t^2} \tag{12}$$

where for tubular cross sections

$$\left(I_1, I_2, I_3\right) = \sum_{k=1}^{N} \pi \rho^{(k)} \left(\left(r_k^2 - r_{k-1}^2\right), 0, \frac{1}{4}\left(r_k^4 - r_{k-1}^4\right) \right) \tag{13}$$

Expressing equations of motion in terms of displacement we have in matrix form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} - \begin{bmatrix} I_1 & 0 & I_2 \\ 0 & -I_1 & 0 \\ I_2 & 0 & I_3 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} = \begin{bmatrix} -p_x \\ p_z \\ 0 \end{bmatrix} \tag{14}$$

where

$$\begin{aligned} & L_{11} = A_{11} \frac{\partial^2}{\partial x^2},\ L_{22} = -A_{55} \frac{\partial^2}{\partial x^2}, L_{33} = D_{11} \frac{\partial^2}{\partial x^2} - A_{55},\ L_{13} = L_{31} = B_{11} \frac{\partial^2}{\partial x^2} \\ & L_{21} = L_{12} = 0,\ L_{23} = L_{32} = -A_{55} \frac{\partial}{\partial x} \end{aligned} \tag{15}$$

In this chapter a numerical method named the General Differential Quadrature (GDQ) is used to solve the equations. The method has been used by several investigators [22-24]. In this method, the derivative of a function is approximated by a weighted linear sum of the function values at all the discrete points.

$$\frac{dF^{(n)}(x_i)}{dx} = \sum_{j=1}^{N} C_{ij}^{(n)} F\left(x_j\right) \tag{16}$$

Generally, by considering the Lagrange interpolation polynomials as test functions, the coefficients for the first order derivative become

$$C_{ij,i\neq j}^{(1)} = \frac{M(x_i)}{(x_i - x_j) M(x_j)} \tag{17}$$

$$M(x_i)=\prod_{j=1,\,j\neq i}^{N}(x_i-x_j) \tag{18}$$

for higher order derivatives

$$C_{ij,i\neq j}^{(n)}=n\left(C_{ij}^{(1)}.C_{ii}^{(n-1)}-C_{ij}^{(n-1)}/\left(x_i-x_j\right)\right) \tag{19}$$

and for all derivatives

$$C_{ii}^{(n)}=-\sum_{j=1,\,j\neq i}^{N}C_{ij}^{(n)} \tag{20}$$

The sampling points are selected based on the Chebyshev–Gauss–Lobatto (C–G–L) grid distribution to get denser population near boundaries.

$$x_i=\frac{a}{2}\left[1-\cos\left(\frac{\pi\left(i-1\right)}{N-1}\right)\right] \tag{21}$$

Boundary conditions

The following boundary conditions can be applied at the ends (x=0, l).

Simply supported (S1): $w=N=M=0$ (22)
Simply supported (S2): $w=u=M=0$ (23)
Clamped (C1): $N=w=\psi=0$ (24)
Clamped (C2): $u=w=\psi=0$ (25)
Free (F1): $N=M=Q=0$ (26)
Free (F2): $u=M=Q=0$ (27)

Simply supported boundary (S1) boundary condition has been considered for case studies in this chapter. The hinge boundary conditions will consist of continuity of displacement functions and shear force as well as zero moment condition. For example in a two-segment shaft with length of l_1 and l_2 and coupling masses m_1 and m_2 (Figure 5.2), the following boundary condition would be applied.

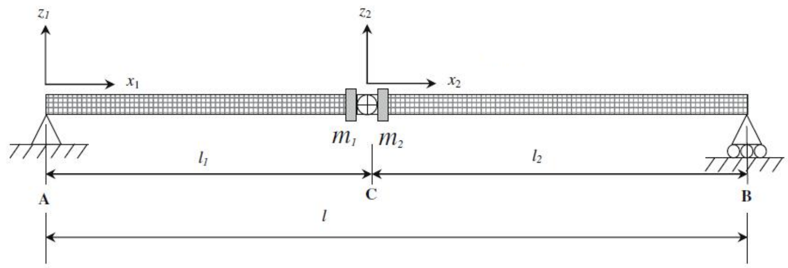

Figure 5.2 Two-segment shaft with hinge connection

$$u_1(l_1,t)=u_2(0,t) \tag{28}$$

$$w_1(l_1,t)=w_2(0,t) \tag{29}$$

$$A_{55}\left(\psi_1(l_1,t)+\frac{\partial w_1(l_1,t)}{\partial x}\right)-m_1\frac{\partial^2 w_1(l_1,t)}{\partial t^2}=\left(\psi_2(0,t)+\frac{\partial w_2(0,t)}{\partial x}\right)+m_2\frac{\partial^2 w_2(0,t)}{\partial t^2} \tag{30}$$

$$B_{11}\frac{\partial u_1(l_1,t)}{\partial x}+D_{11}\frac{\partial \psi_1(l_1,t)}{\partial x}=0 \tag{31}$$

$$B_{11}\frac{\partial u_2(0,t)}{\partial x}+D_{11}\frac{\partial \psi_2(0,t)}{\partial x}=0 \tag{32}$$

For multi-segment shafts these boundary conditions should be applied at any hinge location. Application of GDQ method along with boundary conditions to the equations of motion will give a system of equations in matrix form. The eigenvalues of the system would be natural frequencies that are assessed using MATLAB. The results would then be compared to the published literature and FEM in the next section.

Numerical results

In this section, the present method is compared to different models in the literature in special cases and also to finite element models. A convergence study was first done for the GDQ analysis and it was found that using 11 points in the length yields convergent results for the first 3 significant figures.

In order to verify the model, as the first case the experimental results for a boron/epoxy helicopter driveshaft provided by Zinberg and Symonds [25] is used. A number of researchers have worked on this shaft with different beam and shell models and their results along with the present FSDT and CBT method [8] are shown in Table 5.1. The material and geometrical parameters are given below

E_1 = 211 GPa, E_2 = 24 GPa, $G_{12} = G_{13} = G_{23}$ = 6.9 GPa, ν_{12}= 0.36, density = 1967 kg/m^3), length = 2470 mm, mean diameter = 126.9 mm, thickness = 1.321 mm, laminate [90/45/-45/0_6/90] from inner to outer.

Table 5.1 Boron-epoxy shaft fundamental natural frequencies (Hz) by different authors (E_1 = 211, E_2 = 24, $G_{12} = G_{13} = G_{23}$ = 6.9 GPa, ν_{12}= 0.36, ρ = 1967 kg/m^3), l = 2470 mm, mean diameter = 126.9 mm, thickness = 1.321 mm laminate [90/45/-45/0_6/90]

Author	**Method used**	**Frequency (Hz)**
Zinberg, Symonds [25]	Measured experimentally	91.67
Chang et al. [191]	Continuum based Timoshenko Beam	96.03
Singh and Gupta [54]	Effective Modulus Beam Theory	95.78
Qatu and Iqbal [10]	Finite element analysis using ABAQUS	95.4
Kim and Bert [16]	Sanders shell theory	97.87
	Donnell shallow shell theory	106.65
Bert and Kim [17]	Bresse–Timoshenko beam theory	96.47
	Euler–Bernoulli beam theory	102.47
dos Reis et al. [27]	Bernoulli–Euler beam theory. Stiffness determined by shell finite elements	82.37
Present study	Beam CBT	96.12
	Beam SDBT	90.36

For single-span simply supported shafts, the exact solution using trigonometric functions for displacements, (as proposed in Chapter III) can be used. The results show that most of the models can predict the natural frequencies of this shaft. However, the

FSDT used in this chapter is the most accurate model for this case and predicts the natural frequency with 1 percent error.

The effect of ply orientation on reduction of stiffness and consequently natural frequency of a AS4 graphite-epoxy shaft is provided by Bert and Kim [17]. The shaft has the same geometry as the previous boron/epoxy shaft and the material properties are (E_1=139, E_2 = 11, G_{12} = G_{13} = 6.05, G23 = 3.78 GPa, ν_{12} = 0.313, ρ = 1478 kg/m^3). Their results for the first natural frequency and output of the present model using CBT [8], FSDT and a shell FEM model are presented in Table 5.2.

Table 5.2 Effect of lamination angle on fundamental natural frequencies (Hz) of a AS4-epoxy shaft. (E_1 = 139, E_2 = 11, G_{12} = G_{13} = 6.05, G_{23} = 3.78 GPa, ν_{12} = 0.313, ρ = 1478 kg/m^3)

Theory	Lamination angle						
	0	15	30	45	60	75	90
Sanders Shell [17]	92.12	72.75	50.13	39.77	35.33	33.67	33.28
Bernoulli-Euler [17]	107.08	89.88	71.15	52.85	38.2	31.42	30.22
Bresse-Timoshenko [17]	101.2	86.82	69.95	52.38	37.97	32.9	30.05
Present CBT approach	108.42	71.12	46.05	36.15	32.17	30.78	30.5
Present SDBT approach	100.99	70.01	45.85	36.04	32.07	30.66	30.27
Present FEM analysis	100.28	68.8	45.51	35.9	31.96	30.57	30.27

Results show that the Sanders shell theory and both CBT theories have errors in predicting the natural frequencies. The Bresse-Timoshenko theory only have accurate results in [0] and [90] laminate where bending-twisting coupling is not present. However, the present model while being simple (1D beam model) is the closest to FEM in all cases. It also shows the decrease in the natural frequencies by lowering stiffness and bending twisting coupling.

In order to check the ability of the model in multi-segment shafts, exact solution to the Euler-Bernoulli beam theory for a cross-ply graphite-epoxy shaft conducted by Qatu and Iqbal [10] has been compared to the present method. Figure 5.2 shows the shaft geometry and parameters. The shaft OD was 60 mm and the thickness was 1.5 mm. The material properties are (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3, l=2, OD=0.06, t=0.0015 m). The results of first five natural frequencies for different joint locations are presented in Table 5.3 and match well with those obtained in Ref. [10]. As expected, one can see that the difference between classic and shear deformation model increases with mode number. The effective length (length of the half sine wave of the mode) becomes less for higher modes and the classic model assumption leads to inaccurate results.

The capability of the model for generally laminated two-segment shafts is the next case to be verified. Since such results do not exist in the literature, the authors created a finite element model using shell elements in ANSYS software to validate the model. The geometry and material properties are the same as the previous cross-ply shaft. In the finite element model, the shaft is created using Shell99 composite shell element and the joint is created using MPC184 element. The results of the first natural frequency for different coupling locations with bending-twisting coupled and bending-stretching coupled laminates are presented in Table 5.4 and 5.5. The geometry and material properties are the same as the previous cross-ply shaft in Table 5.4. But the diameter is reduced to 20 mm in Table 5.5 to see if decreasing the radius affects the method. The results show a high level of accuracy for different laminates. The results show a high level of accuracy and the error is not more than 1 percent in almost all cases.

Table 5.3 Comparison of natural frequencies of a simply supported two-segment graphite epoxy shaft (Hz) with Ref[10] (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, $\upsilon12$=0.3, ρ =1580 kg/m^3, l=2, OD=0.06, t=0.0015 m)

Ref[10]					Present				
both [0]									
l_1/l	Mode I	II	III	IV	l_1/l	I	II	III	IV
0.1	138.2	450.9	945	1619.8	0.1	133.3	405.4	771.8	1186.6
0.2	164.8	544.4	1143.2	1906.7	0.2	158.3	479.9	897.8	1328.6
0.3	203.6	665.3	1072.6	1576.8	0.3	193.6	568.3	864.1	1177.8
0.4	258.5	594.1	989.1	1906.7	0.4	242.1	520.2	808.9	1329.2
0.5	305.0	476.5	1220.2	1544.2	0.5	281.1	432.7	940.9	1167.6
both [90]									
l_1/l	Mode I	II	III	IV	l_1/l	I	II	III	IV
0.1	35.1	114.6	240.2	411.8	0.1	34.8	112.8	233.9	394.9
0.2	41.9	138.4	290.6	484.7	0.2	41.6	136.2	282.2	462.8
0.3	51.7	169.1	272.7	400.8	0.3	51.4	166.1	265.2	385.5
0.4	65.7	151	251.4	484.7	0.4	65.1	148.5	245.0	463.0
0.5	77.5	121.1	310.2	392.5	0.5	76.8	119.5	300.8	377.8
1st [0], 2nd [90]									
l_1/l	Mode I	II	III	IV	l_1/l	I	II	III	IV
0.1	35.3	114.5	240.5	412.8	0.1	35.0	112.8	234.0	395.2
0.2	51.9	128.8	294.1	508.1	0.2	41.7	136.6	285.3	483.3
0.3	51.9	175.2	374.2	646.3	0.3	51.5	171.9	360.0	604.8
0.4	67.3	230.9	483.6	667.8	0.4	66.7	225.1	455.4	596.6
0.5	92.2	309.1	443.4	741.6	0.5	91.1	296.2	410.1	688.1

Table 5.4 Comparison of first natural frequencies for simply supported two-segment bending-stretching coupled graphite-epoxy shaft with FEM (Hz) (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3, l=2, OD=0.06, t=0.0015 m)

Laminate	[0/90/0/90]		[90/90/0/0]	
l_1/l	Present	FEM	Present	FEM
0.1	32.628	33.556	29.962	31.06
0.2	38.884	39.706	35.757	36.699
0.3	47.962	48.826	44.152	45.102
0.4	60.853	61.796	56.029	57.065
0.5	71.693	72.776	66.412	67.192

Table 5.5 Natural frequencies comparison of bending-twisting coupled laminated simply supported two-segment graphite-epoxy shaft with FEM (Hz) (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3, l=2, OD=0.02, t=0.0015 m)

l_1/l	Laminate	0	15	30	45	60	75	90
0.1	Present	133.3	92.90	60.56	45.99	39.02	35.78	34.80
	FEM	133.4	93.43	61.10	46.36	39.23	35.87	34.88
0.2	Present	158.3	110.7	72.29	54.94	46.62	42.75	41.60
	FEM	158.3	111.2	72.91	55.34	46.84	42.84	41.65
0.3	Present	193.6	136.1	89.10	67.76	57.52	52.76	51.40
	FEM	193.0	136.4	89.79	68.24	57.77	52.85	51.40
0.4	Present	242.1	171.5	112.8	85.87	72.94	66.91	65.10
	FEM	239.7	169.5	113.4	86.43	73.23	67.01	65.17
0.5	Present	281.1	200.7	132.7	101.2	86.00	78.90	76.80
	FEM	281.5	200.7	133.7	101.9	86.35	79.02	76.85
Avg. error (%)		-0.22	0.02	0.77	0.72	0.45	0.19	0.10

As the last case, the effect of coupling mass on bending natural frequencies of the previous shaft is presented at Table 5.6. It shows that the fundamental frequency is reduced by 23% for length ratio of 0.2 and 14% for length ratio of 0.2 when 1 kg of combined mass is present. The lumped mass does not have any effect on the fourth mode because a nodal line in that mode passes through these masses.

Table 5.6 Natural frequencies (Hz) of two-segment graphite-epoxy shaft with different lumped masses and [0] laminate in both shafts (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3, l=2, OD=0.06, t=0.0015 m)

l_1/l =0.2					l_1/l =0.4				
m_1+m_2	Mode I	II	III	IV	m_1+m_2	I	II	III	IV
0	158.3	479.9	897.8	1328.6	0	242.1	520.2	808.9	1329.2
0.25	133.5	442.3	865.2	1328.6	0.25	221.1	461.3	726.3	1329.2
0.5	126.7	434.4	858.2	1328.6	0.5	213.9	446.1	713.5	1329.2
0.75	123.6	431.0	855.2	1328.6	0.75	210.3	439.4	708.5	1329.2
1	121.7	429.1	853.5	1328.6	1	208.1	435.7	705.9	1329.2

Conclusion

A modified FSDT model that accounts for multi-segments, lumped mass, different laminates, shear deformation and rotary inertia was validated for transverse vibration analysis of composite shafts. The proposed model uses lamina modulus for calculation of ABD parameters. The method was verified using results in the literature and finite element models. The results showed good accuracy of the model for multi-segment shafts in transverse vibration analyses for all kinds of laminates. This model provides an accurate set of equations for calculating the natural frequencies of composite multi-segment shafts with lumped mass and arbitrary laminate.

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CHAPTER VI

CONCLUDING REMARKS

A modified FSDT model was proposed for static and vibration analysis of composite beams and shafts. It has been shown that using one dimensional modulus of elasticity of each ply for calculation of A_{11}, B_{11}, and D_{11}, one can perform accurate analysis of static and vibration behavior for composite beams and shafts with any laminate. The results were compared to 3D FEM and FSDT has been shown to be accurate enough for the analysis and GDQ method to be accurate for analysis of different boundary conditions.

The deepness term is exactly integrated into A_{11}, B_{11}, and D_{11}, equations and it is concluded that using modulus of elasticity along with deep formulation for stiffness parameters, the FSDT model can provide accurate analysis for curved beams. The results compared to those obtained by a converged FEM 3D model, showed good accuracy. The model has also been compared to TSDT and it is shown that TSDT using modified ABDs is not accurate for asymmetric laminates. It increases the labor but it does not add accuracy for vibration analysis.

The modified stiffness parameters for shafts has been proposed and used for transverse vibration analysis of multi-segments composite shafts with lumped mass. The model accounted for multi-segments, lumped mass, different laminates, shear deformation and rotary inertia. The proposed FSDT model was verified against results in the literature and finite element models. The results showed good accuracy of the model for multi-segment shafts in transverse vibration analyses for all kinds of laminates.

The model proposed in this thesis, provides an accurate set of equations for analysis of different types of beams (straight and curved) and shafts with arbitrary laminate for engineers and scientists. It can be expanded to treat shafts with lumped masses (e.g. gears) and other complexities. As a future work, an element can be developed based on the FSDT model in this thesis in order to solve the problem of beams with varying cross section, tapered or complicated geometry.

APPENDIX A

NONDIMENSIONAL MAXIMUM DEFLECTION, MOMENT AND NATURAL FREQUENCIES OF SIMPLY SUPPORTED CURVED BEAM WITH a/h=10

Table A.1 Nondimensional maximum deflection and moment of simply supported curved beam with a/h=10 (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3)

a/R	Method	[0]		$[0/90]_s$		[0/90]		[45]		[30/60]	
		w	M	w	M	w	M	w	M	w	M
0.2	FSDT (E_x)	2.245	3.140	2.455	3.140	7.662	3.140	14.65	3.140	13.68	3.140
	3D FEM	2.339	3.304	2.553	3.092	7.872	3.240	14.42	3.223	13.46	3.160
0.6	FSDT (E_x)	2.396	3.248	2.620	3.248	8.036	3.248	15.64	3.248	14.51	3.248
	3D FEM	2.520	3.425	2.747	3.210	8.311	3.360	15.42	3.337	14.31	3.295
1	FSDT (E_x)	2.747	3.489	3.004	3.489	9.060	3.489	17.95	3.489	16.55	3.489
	3D FEM	2.939	0.000	3.197	3.420	9.46	3.580	17.75	3.589	16.42	3.519
2	FSDT (E_x)	6.205	5.319	6.786	5.319	19.63	5.319	40.69	5.319	36.94	5.319
	3D FEM	7.096	5.735	7.657	5.274	21.25	5.52	40.78	5.505	37.32	5.415

Table A.2 Nondimensional natural frequencies ($\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}$) of simply supported curved beam with a/h=10 (E_1 = 138, E_2 =8.96, G_{12}= 7.1 GPa, υ_{12}=0.3, ρ =1580 kg/m^3)

$[0]_4$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	8.286	7.999	8.136	7.847	7.892	7.599
2	24.46	22.63	24.35	22.52	24.17	22.34
3	41.48	37.48	41.39	37.40	41.25	37.27
4	58.21	52.03	58.14	51.97	58.03	51.87
5	74.66	66.94	74.60	66.89	74.50	66.81
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	7.559	7.262	7.145	6.845	4.194	3.923
2	23.91	22.08	23.59	21.76	21.00	19.21
3	41.06	37.09	40.81	36.85	38.76	34.94
4	57.87	51.73	57.67	51.55	56.00	50.08
5	74.37	66.69	74.20	66.54	72.79	65.30
$[0/90]_s$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	7.918	7.655	7.775	7.511	7.541	7.276
2	23.85	22.01	23.74	21.90	23.56	21.73
3	40.87	36.65	40.79	36.58	40.65	36.45
4	57.68	51.06	57.61	51.00	57.50	50.90
5	74.19	65.95	74.13	65.90	74.04	65.82
	a/R=0.8		**a/R=1**		**a/R=2**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	7.223	6.956	6.828	6.560	4.009	3.775
2	23.31	21.48	23.00	21.18	20.47	18.72
3	40.46	36.27	40.21	36.05	38.20	34.20
4	57.34	50.76	57.14	50.58	55.49	49.13
5	73.91	65.70	73.74	65.54	72.34	64.29
$[0_2/90_2]$						
	a/R=0.2		**a/R=0.4**		**a/R=0.6**	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	4.468	4.375	4.425	4.328	4.327	4.227
2	15.92	14.96	15.96	14.97	15.95	14.95
3	30.94	27.97	31.03	28.03	31.09	27.98
4	47.36	41.64	47.49	41.72	47.58	41.77
5	64.23	55.27	64.37	55.35	64.48	55.41

Table A.2 Continued

[0$_2$/90$_2$]						
	a/R=0.8		a/R=1		a/R=2	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	4.177	4.074	3.977	3.872	2.395	2.302
2	15.89	14.87	15.78	14.75	14.52	13.44
3	31.11	28.04	31.08	27.98	30.31	27.13
4	47.64	41.79	47.67	41.78	47.23	41.27
5	64.56	55.45	64.61	55.48	64.36	55.27
[45]$_4$						
	a/R=0.2		a/R=0.4		a/R=0.6	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	3.221	3.226	3.164	3.168	3.069	3.073
2	12.09	11.90	12.03	11.85	11.94	11.64
3	24.77	23.69	24.72	23.65	24.64	23.59
4	39.66	38.22	39.62	38.19	39.55	38.15
5	55.71	51.99	55.68	51.96	55.62	51.92
	a/R=0.8		a/R=1		a/R=2	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	2.940	2.943	2.780	2.782	1.634	1.632
2	11.82	11.64	11.67	11.49	10.42	10.31
3	24.54	23.50	24.40	23.38	23.25	22.37
4	39.46	38.09	39.33	38.02	38.31	37.61
5	55.53	51.85	55.42	51.77	54.50	51.09
[30$_2$/60$_2$]						
	a/R=0.2		a/R=0.4		a/R=0.6	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	3.338	3.345	3.290	3.294	3.202	3.205
2	12.46	12.25	12.44	12.23	12.39	12.18
3	25.39	24.22	25.41	24.23	25.39	24.22
4	40.46	38.80	40.50	38.84	40.51	38.87
5	56.61	52.65	56.66	52.69	56.69	52.72
	a/R=0.8		a/R=1		a/R=2	
n	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM	FSDT (E_x)	3D FEM
1	3.077	3.077	2.918	2.916	1.730	1.720
2	12.30	12.08	12.17	11.96	11.02	10.91
3	25.34	24.17	25.26	24.10	24.37	23.27
4	40.50	38.89	40.45	38.89	39.79	38.79
5	56.70	52.72	56.68	52.71	56.19	52.40